LECTURE 1 Introduction to Mechanics of Materials. Geometrical Properties of Cross Sections of a Rod (Part 1)

1 Problems and Methods in Mechanics of Materials (syn. Strength of Materials)

Mechanics of materials is the science of **strength**, **stiffness** and **stability** of elements of engineering structures.

Strength is understood as the ability of structure or its elements to withstand a specified external loading without fracture.

Stiffness (or rigidity) is understood as the capability of a body or structural element to resist deformation i.e. to prevent exceeding elongations, deflections and so on.

Stability is meant as the capability of a structure to resist the forces which tend to move it from the initial state of equilibrium i.e. to prevent **buckling**.

Mechanics of materials is one of the branches of **mechanics of deformable solids**. Mechanics of deformable solids includes also other branches such as the mathematical **theory of elasticity, theory of plates and shells (structural mechanics)**. In contrast, classical (theoretical) mechanics deals with **nondeformable** solids and the problems of their equilibrium and movement.

The mathematical theory of elasticity studies the behavior of deformable solids under external mechanical and thermal loading using a complex mathematical apparatus. Mechanics of materials uses a simple mathematical apparatus and simplifying hypotheses for strength, rigidity and stability analysis. It performs simple approximate calculations of typical structural elements from the viewpoint of their strength, rigidity and stability.

The general goal of engineering design is to prevent the structure failure (**design against failure**). The structure is not able to work at the level of fracture, i.e. should not fail under applied **external loads**. It must have preliminary grounded **factor of safety**. The lack of factor leads to **fracture**, but insufficient factor makes structure imperfect. The correct choice of the factor is a responsible problem in mechanical engineering.

The geometrical scheme in strength of materials is the scheme of a **rod**. A rod generally implies a body one of whose dimensions (length) is considerably greater than the other two. **Bars**, **beams**, **shafts**, **shells** are also considered in mechanics of materials.

The concepts of **displacement, deformation and stress** are of the most importants in mechanics of materials.

2 Geometrical Properties of Cross Sections of a Rod

In solving the problems in mechanics of materials, it is necessary to operate with some geometrical properties of cross sections of a rod which influence on ability of engineering structure to withstand applied load. Simplest example of a rod processing by welding is shown on Fig. 1.

2.1 Cross-Section Area

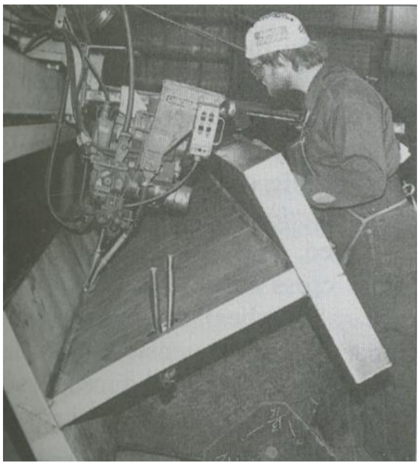


Fig. 1

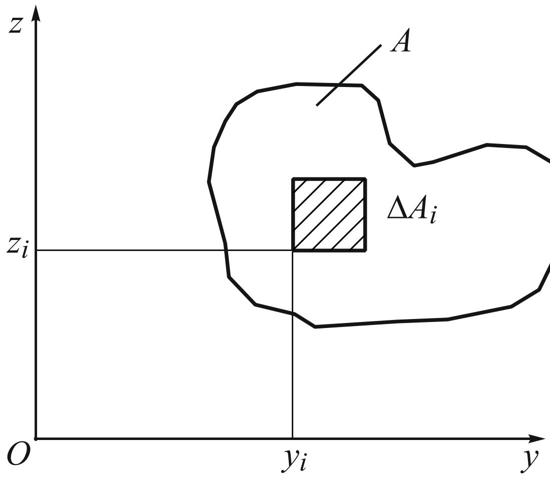


Fig. 2

Take a **cross section** of a rod. Relate it to a system of coordinates y, z. Isolate an element ΔA from the area A with coordinates y, z. Consider the following integral:

$$A = \lim_{\Delta A_i \to 0} \sum_{1}^{\infty} \Delta A_i = \int_A dA, \qquad (1)$$

where the index A beneath the integral sign indicates that the integration is carried out over the whole cross-sectional area. The integral (1) is called as **cross-section area**.

Cross-sectional areas of simple figures

Circle

$A = \pi r^2 = \dfrac{\pi d^2}{4}$ – area.

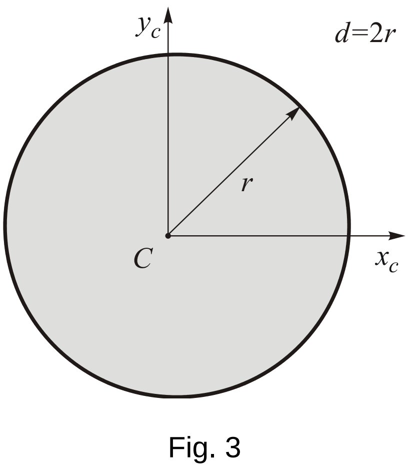


Fig. 3

Circular sector

α =angle in radians, $(\alpha \le \pi/2)$,

$A = \alpha r^2$ – area.

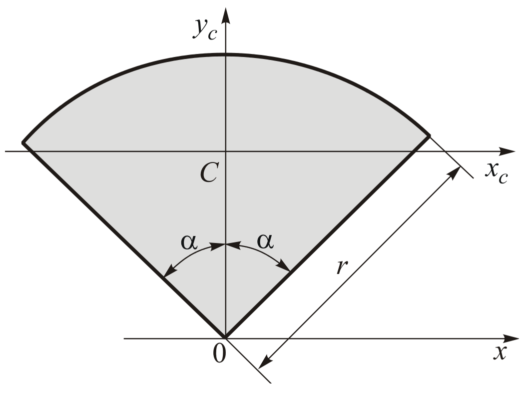


Fig. 4

Circular segment

Origin of axes at center of circle,
α =angle in radians, $(\alpha \le \pi/2)$,
$A = r^2(\alpha - \sin\alpha\cos\alpha)$ – area.

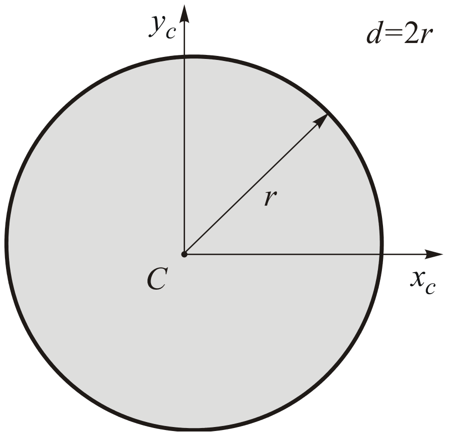


Fig. 5

Circle with core removed

α =angle in radians, $(\alpha \le \pi/2)$,
$\alpha = \arccos\frac{a}{r}$, $b = \sqrt{r^2 - a^2}$;
$A = 2r^2\left(\alpha - \frac{ab}{r^2}\right)$ – area.

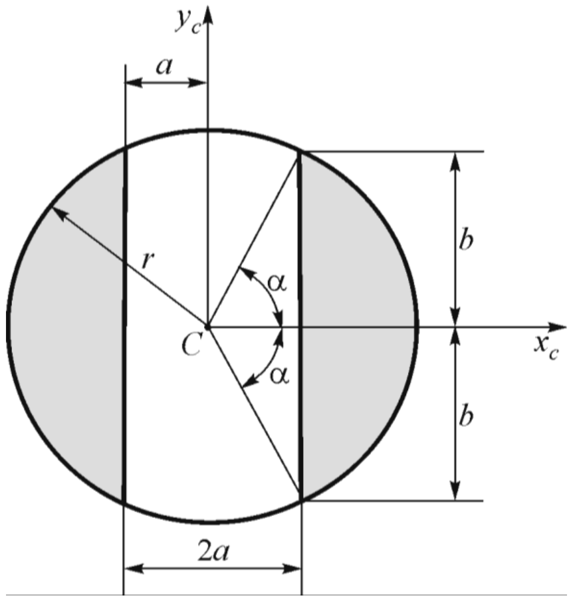


Fig. 6

Circular segment

Origin of axes at center of circle,
α =angle in radians, $(\alpha \le \pi/2)$,
$A = r^2(\alpha - \sin\alpha\cos\alpha)$ – area.

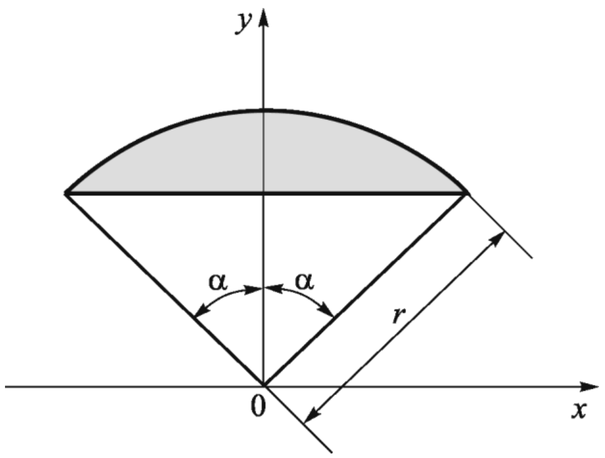


Fig. 7

Equilateral triangle

a – side,
$A = \frac{1}{4}a^2\sqrt{3}$ – area.

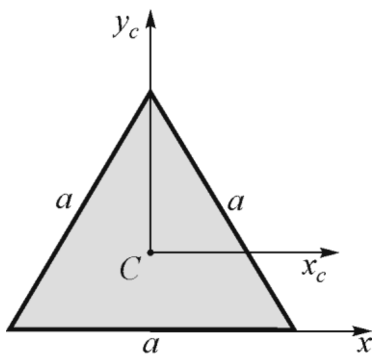


Fig. 8

Ellipse

Origin of axes at centroid,
$A = \pi ab$,
a – magor axis,
b – minor axis;
Circumference ≈

$$\approx \pi\left[1.5(a+b)-\sqrt{ab}\right](a/3 \le b \le a) \approx$$

$$\approx 4.17b^2/a + 4a(0 \le b \le a/3).$$

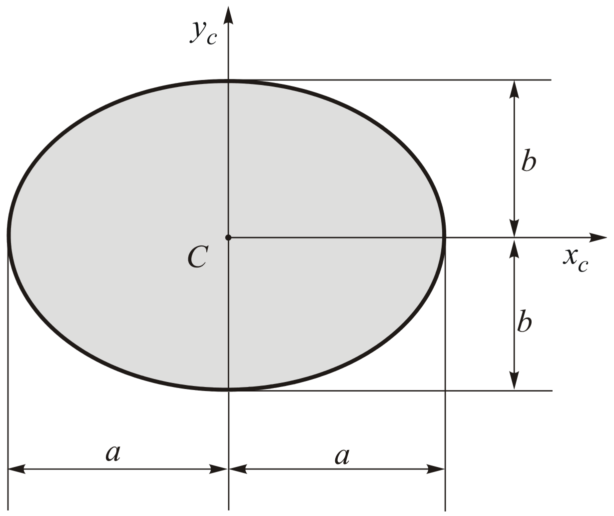


Fig. 9

Hollow circular cross section

$$A = \pi\left(r_2^2 - r_1^2\right),$$

r_1 – inner radius,
r_2 – outer radius,
$t = r_2 - r_1$ – thickness.

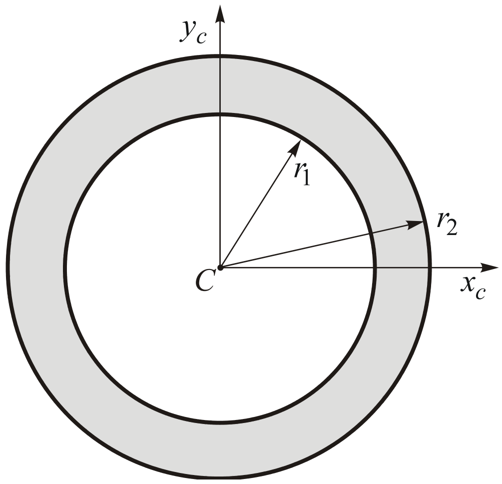


Fig. 10

Hollow square cross section (doubly symmetric)

$A = b^2 - c^2$ – area,
C – centroid.

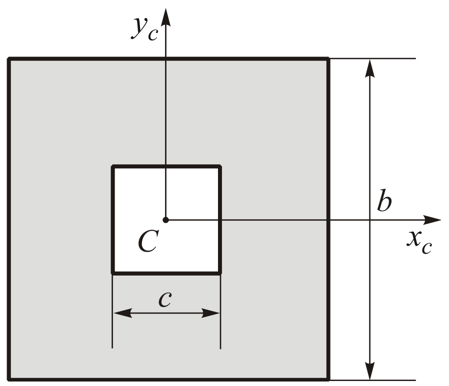


Fig. 11

Isosceles trapezoid

$$A = \frac{h(b_1+b_2)}{2} \text{ – area,}$$

C – centroid, h – height.

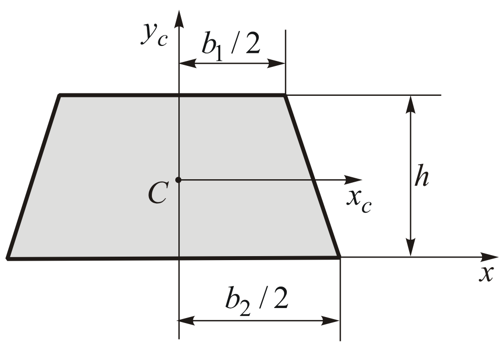


Fig. 12

Isosceles right triangle

$$A=\frac{b^2}{4}\ \text{– area, } C \text{ – centroid.}$$

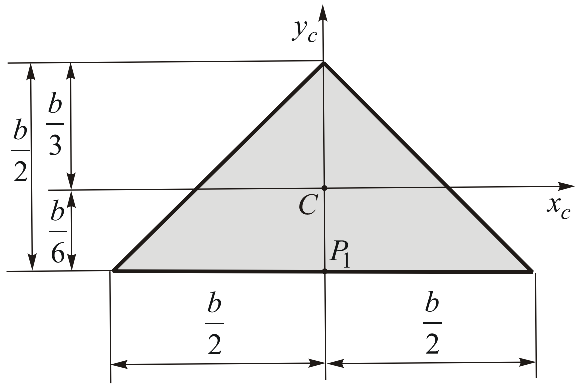


Fig. 13

Isosceles triangle

$$A=bh/2\ \text{– area,}$$
$$h \text{ – height, } b \text{ – width.}$$

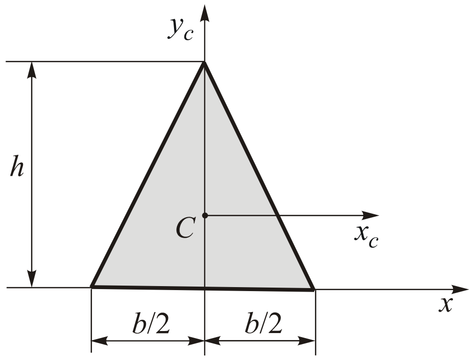


Fig. 14

Parabolic semisegment

$$y=f(x)=h\left(1-\frac{x^2}{b^2}\right),\ A=\frac{2bh}{3}\ \text{– area.}$$

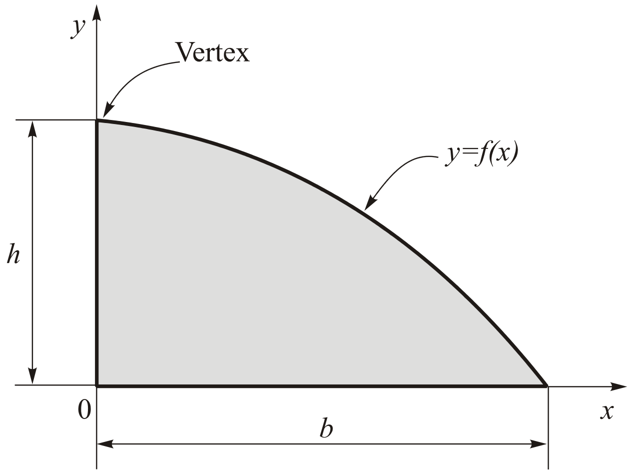


Fig. 15

Parabolic spandrel

$$y=f(x)=\frac{hx^2}{b^2},\ A=\frac{bh}{3}\ \text{– area.}$$

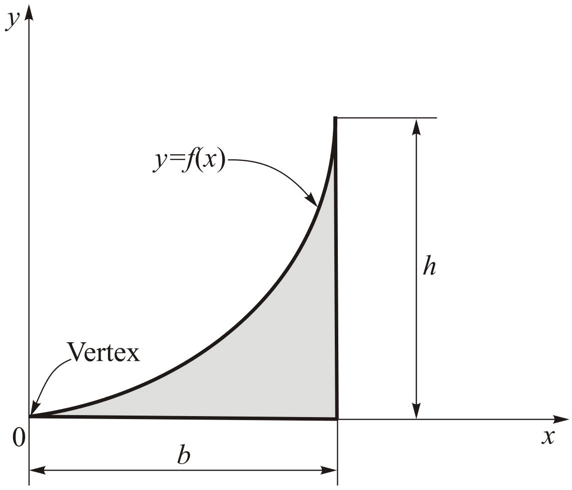


Fig. 16

Rectangle

$$A=bh\ \text{– area.}$$

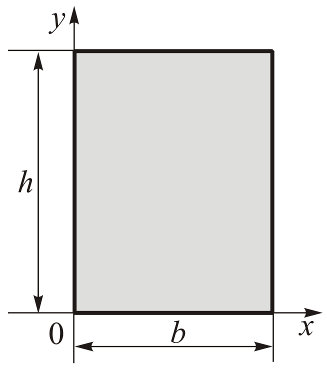


Fig. 17

Right triangle

$$A=bh/2\ \text{– area.}$$

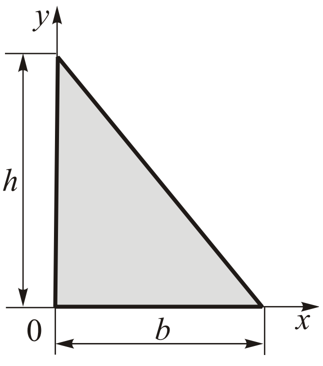


Fig. 18

Quarter circle

$$A=\frac{\pi r^2}{4} \text{ – area.}$$

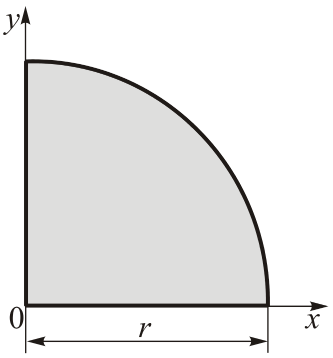


Fig. 19

Quarter-circular spandrel

$$A=\left(1-\frac{\pi}{4}\right)r^2 \text{ – area.}$$

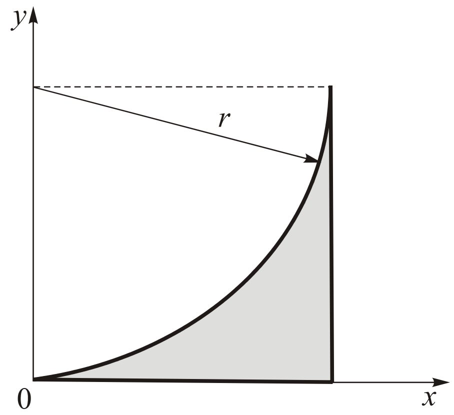


Fig. 20

Regular hexagon

b – side, C – centroid, $A=\frac{3\sqrt{3}}{2}b^2$ – area.

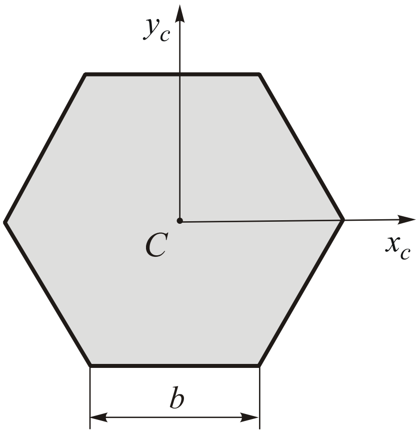


Fig. 21

Regular hexagon hollow cross section (syn. regular hexagon tube)

t – thickness, $A=6bt$ – area.

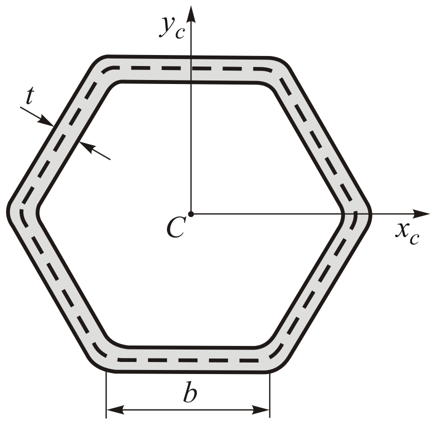


Fig. 22

Semicircle

r – radius,

$$A=\frac{\pi r^2}{2} \text{ – area.}$$

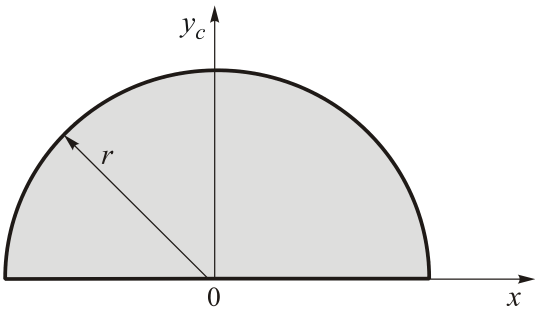


Fig. 23

Sine wave

$$A=\frac{4bh}{\pi} \text{ – area.}$$

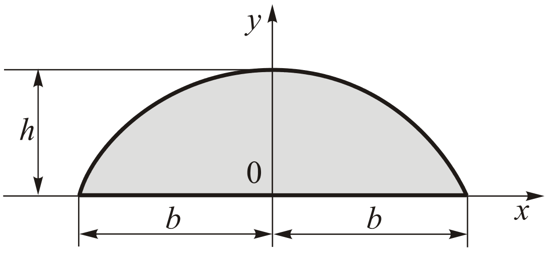


Fig. 24

Regular polygon with *n* sides

n – number of sides $(n \geq 3)$,
b – length of a side,
β – central angle for a side,
α – interior angle (or vertex angle).

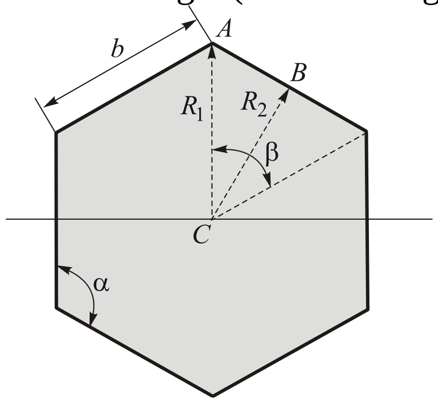


Fig. 25

Semisegment of *n*th degree

$$y = f(x) = h\left(1 - x^n / b^n\right), (n > 0);$$

$$A = bh\left(n / n + 1\right) \text{ – area.}$$

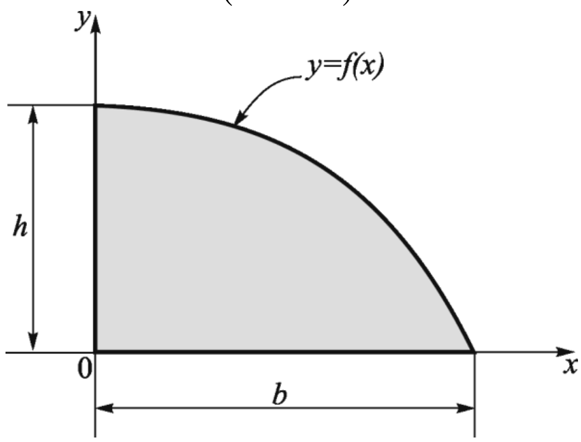


Fig. 26

Square chimney

$$A = b^2 - \pi d^2 / 4 \text{ – area.}$$

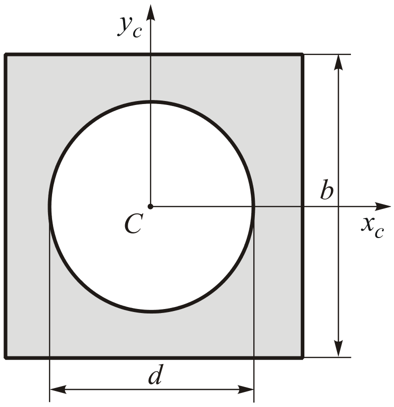


Fig. 27

Square cross section, square

$$A = a^2 \text{ – area.}$$

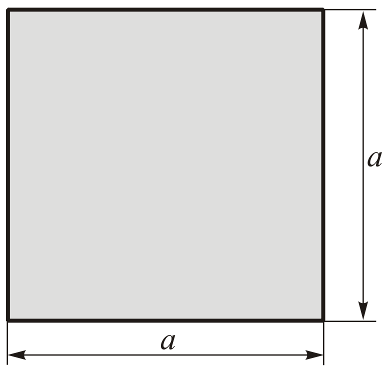


Fig. 28

Square tubular cross section

b – width, t – thickness,
$A = 4bt$ – area.

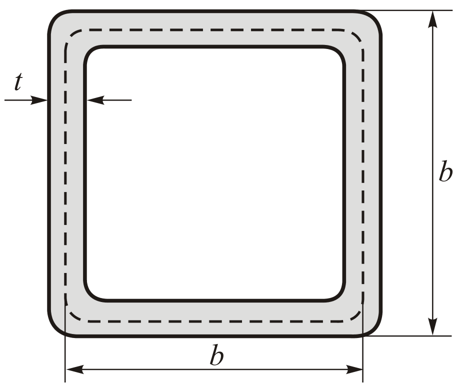


Fig. 29

Thin circular ring

$$A = 2\pi rt = \pi dt,$$

$$d = 2r, \ (t << r).$$

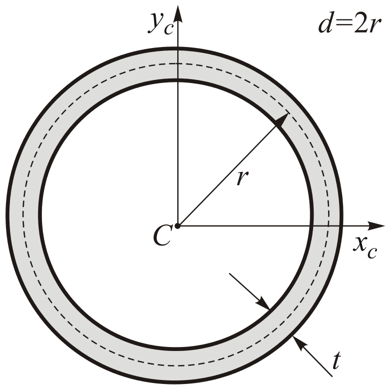


Fig. 30

Triangle

$A = \frac{bh}{2}$ – area.

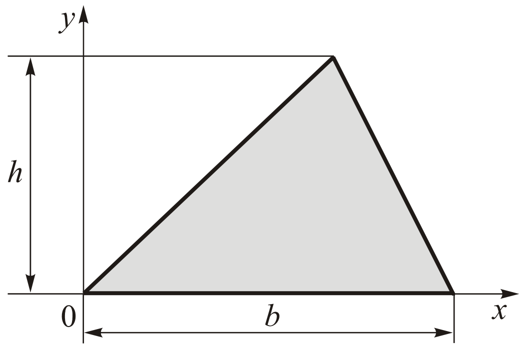


Fig. 31

Trapezoid

$A = \frac{h(a+b)}{2}$ – area.

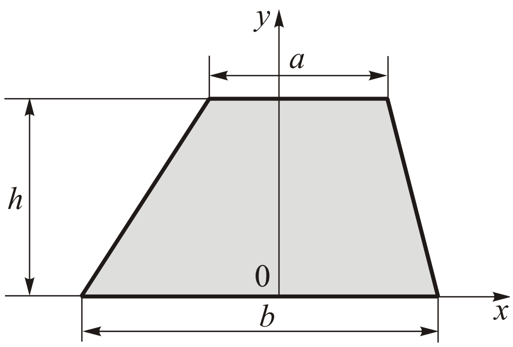


Fig. 32

2.2 Static Moment (First Moment) of a Section

Consider the following two integrals:

$$S_y = \int_A z dA, \qquad S_z = \int_A y dA. \tag{2}$$

Each of them represents the sum of the products of elements area and the distance to respective axis (*y* or *z*).

The first integral is called the **static moment of the section** with respect to the *y* axis, and the second – to the *z* axis.

The static moment is measured in meter cubed (m^3).

According to expressions (2) the static moment can be positive, negative or equal to zero. ***The static moment of a compound section equals to the sum of the static moments of the simplest figures (components).***

2.3 Central Axes. Centroid

Consider a plane section and draw two pairs of parallel axes *y*, *z*, and y_1, z_1 as shown on Fig. 33. Let a distance between the axes will be *b* and *a*. Let us assume that the area *A* of this section and S_y and S_z are given. It is necessary to find the static moments with respect to the y_1 and z_1 axes, i.e. S_{y_1} and S_{z_1}.

According to formulas (2) the static moments are

$$S_{y_1} = \int_A z_1 dA, \qquad S_{z_1} = \int_A y_1 dA. \tag{3}$$

As may be seen from Fig. 33

$$z_1 = z - b, \qquad y_1 = y - a. \tag{4}$$

Substituting y_1 and z_1 from expressions (4) to formulas (3), we find

$$S_{y_1} = \int_A (z-b)dA = \int_A zdA - b\int_A dA,$$

$$S_{z_1} = \int_A (y-a)dA = \int_A ydA - a\int_A dA. \tag{5}$$

Because, as we know

$$\int_A zdA = S_y, \qquad \int_A ydA = S_z, \qquad \int_A dA = A, \tag{6}$$

then we rewrite (5) as

$$S_{y_1} = S_y - bA, \qquad S_{z_1} = S_z - aA. \tag{7}$$

Consider the first of the expressions derived above:

$$S_{y_1} = S_y - bA.$$

The quantity b may be any number whatever, either positive or negative. It can, therefore, always be chosen to make the product bA equal to S_y. Then the static moment with respect to the y_1-axis vanishes, that is

$$0 = S_y - bA, \qquad 0 = S_z - aA. \tag{8}$$

***An axis with respect to which the static moment is zero is called* central axis *or* centroidal axis. *The point of intersection of central axes is called* the center of gravity, or centroid of cross-section.**

Thus, equations (8) make it possible to determine the position of the centroid if the static moments are known:

$$b = Z_c = \frac{S_y}{A}, \qquad a = Y_c = \frac{S_z}{A}. \tag{9}$$

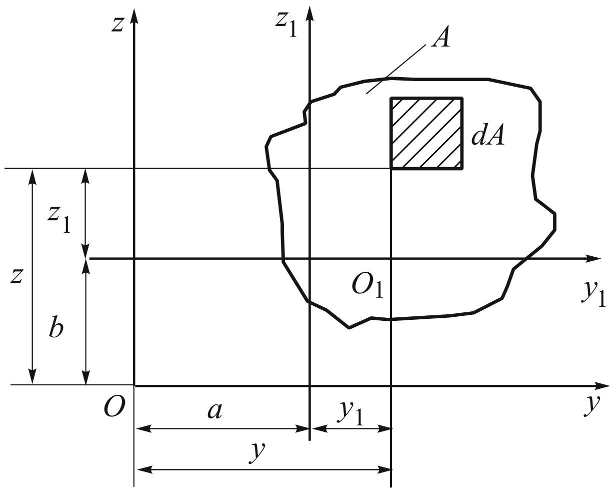


Fig. 33

where Z_c and Y_c **are coordinates of the centroid**. It is possible also to find the static moments if the position of the centroid is known.

The centroid of a composite section is determined by

$$Y_c = \frac{\sum_1^n A_i \cdot z_i}{\sum_1^n A_i}, \quad Z_c = \frac{\sum_1^n A_i \cdot y_i}{\sum_1^n A_i}, \qquad (10)$$

where y_i and z_i are coordinates of the geometrical centers of the component figures.

Consider the simplest example.

Example 1 The calculation of centroid coordinate of the triangular (Fig. 34)

Given: b is the base of the triangle, h is the height.

R.D.: distance of the centroid of the triangle from its base, i.e. z_c.

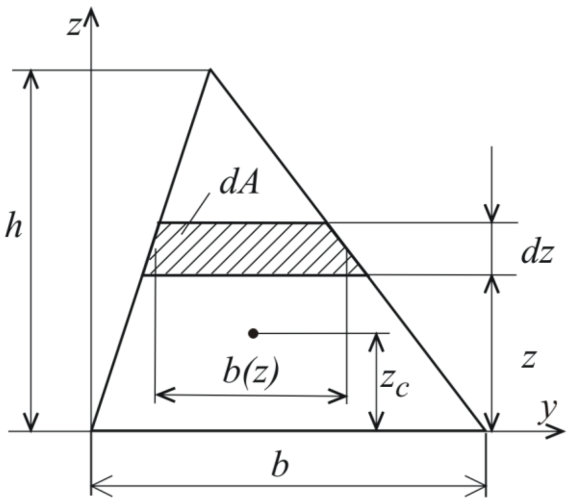


Fig. 34

Solution By definition $z_c = \frac{S_y}{A}$. The triangle static moment with respect to the y axis is equal to

$$S_y = \int_A z dA.$$

In our case, $dA = b(z)dz$, $A = \frac{1}{2}bh$.

From triangles similarity, $b(z)$ equals to

$$b(z) = (h - z)\frac{b}{h}.$$

Thus

$$S_y = \int_0^h z(h - z)\frac{h}{b}dz = \frac{bh^2}{6}, \qquad z_c = \frac{+\frac{bh^2}{6}}{\frac{bh}{2}} = +\frac{h}{3}.$$

The coordinate from the base of the triangle to the centroid of gravity is $h/3$ (up directed) and the centroidal horizontal central axis is located upwards at distance $1/3$ of altitude from its base.

Example 2 Centroidal axes of right triangle (Fig. 35)

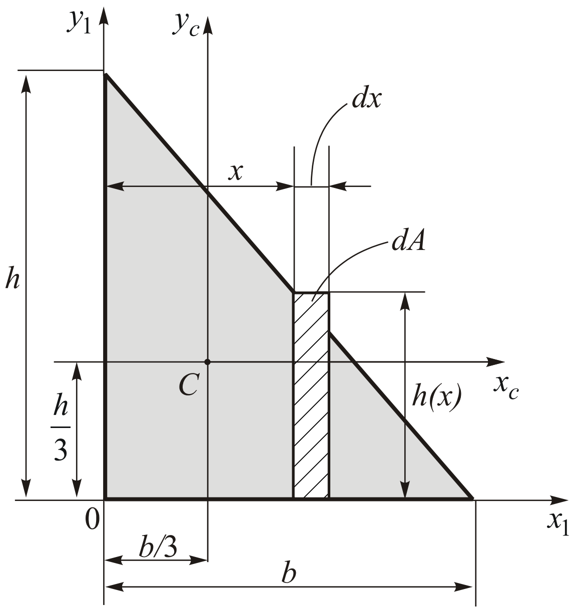


Fig. 35

Given: b is the base of the triangle, h is the height.

R.D.: distance of the centroid of the triangle from its base, i.e. x_c and y_c.

$$x_c = \frac{S_{y_1}}{A} = \frac{\int_0^b x dA}{A} =$$

$$= \left\{ \begin{array}{l} dA = h(x)dx; \\ \frac{h(x)}{h} = \frac{(b-x)}{b} \rightarrow h(x) = \frac{h}{b}(b-x) \end{array} \right\} =$$

$$= \frac{\frac{h}{b}\int_0^b (b-x)xdx}{\frac{bh}{2}} = \frac{\frac{h}{b}\left(b\frac{b^2}{2} - \frac{b^3}{3}\right)}{\frac{bh}{2}} = \frac{\frac{hb^2}{6}}{\frac{bh}{2}} = \frac{b}{3}.$$

By analogy $y_c = \frac{S_{x_1}}{A} = \frac{h}{3}$.

In result, x_c, y_c axes are centroidal axes of right triangle.

Example 3 Centroid of a composite area (Fig. 36)

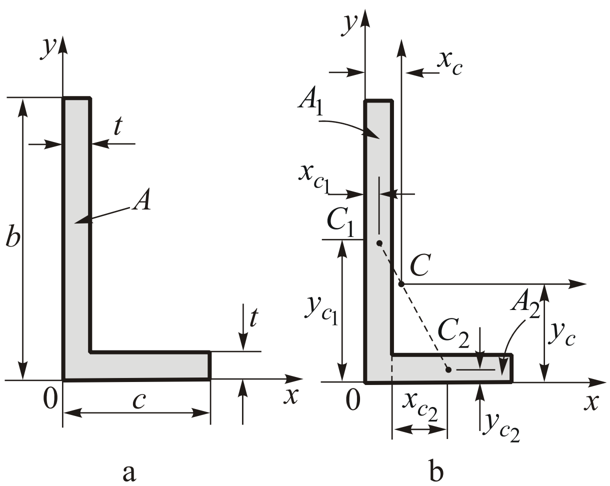


Fig. 36

Given: dimensions of an angular section.

R.D.: x_c, y_c coordinates.

Solution The areas and first moments of composite areas may be calculated by summing the corresponding properties of the component parts. Let us assume that a composite area is divided into a total of n

parts, and let us denote the area of the i th part as A_i. Then we can obtain the area and first moments by the following summations:

$$A = \sum_{i=1}^{n} A_i \,, \tag{1}$$

$$S_x = \sum_{i=1}^{n} y_{c_i} A_i \,, \qquad S_y = \sum_{i=1}^{n} x_{c_i} A_i \,; \tag{2}$$

in which x_{c_i} and y_{c_i} are the coordinates of the centroid of the i part. The coordinates of the centroid of the composite area are

$$x_c = \frac{S_y}{A} = \frac{\sum_{i=1}^{n} x_{c_i} A_i}{\sum_{i=1}^{n} A_i}, \qquad y_c = \frac{S_x}{A} = \frac{\sum_{i=1}^{n} y_{c_i} A_i}{\sum_{i=1}^{n} A_i}. \tag{3}$$

Since the composite area is represented exactly by the n parts, the preceding equations give exact results for the coordinates of the centroid. To illustrate the use of Eq. (3), consider the L-shaped area (or angle section) shown in Fig. 36, a. This area has side dimensions b and c and thickness t. The area can be divided into two rectangles of areas A_1 and A_2 with centroids C_1 and C_2, respectively (Fig. 36, b). The areas and centroidal coordinates of these two parts are

$$A_1 = bt\,, \quad x_{c_1} = \frac{t}{2}\,, \quad y_{c_1} = \frac{b}{2}\,;$$

$$A_2 = (c - t)\,t\,, \quad x_{c_2} = \frac{c - t}{2}\,, \quad y_{c_2} = \frac{t}{2}\,.$$

Therefore, the area and first moments of the composite area (from Eqs. (1) and (2)) are

$$A = A_1 + A_2 = t\left(b + c - t\right),$$

$$S_x = y_{c_1} A_1 + y_{c_2} A_2 = \frac{t}{2}\left(b^2 + ct - t^2\right),$$

$$S_y = x_{c_1} A_1 + x_{c_2} A_2 = \frac{t}{2}\left(bt + c^2 - t^2\right).$$

Finally, we can obtain the coordinates x_c and y_c of the centroid C of the composite area (Fig. 36, b) from Eq. (3):

$$x_c = \frac{S_y}{A} = \frac{bt + c^2 - t^2}{2(b+c-t)}, \quad y_c = \frac{S_x}{A} = \frac{b^2 + ct - t^2}{2(b+c-t)}. \tag{4}$$

Note 1: When a composite area is divided into only two parts, the centroid C of the entire area lies on the line joining the centroids C_1 and C_2 of the two parts (as shown in Fig. 1b for the L-shaped area).

Note 2: When using the formulas for composite areas (Eqs. (1), (2) and (3)), we can handle the absence of an area by subtraction. This procedure is useful when there are cutouts or holes in the figure.

Example 4 Determination the coordinates of the centroid of the compound section (Fig. 37)

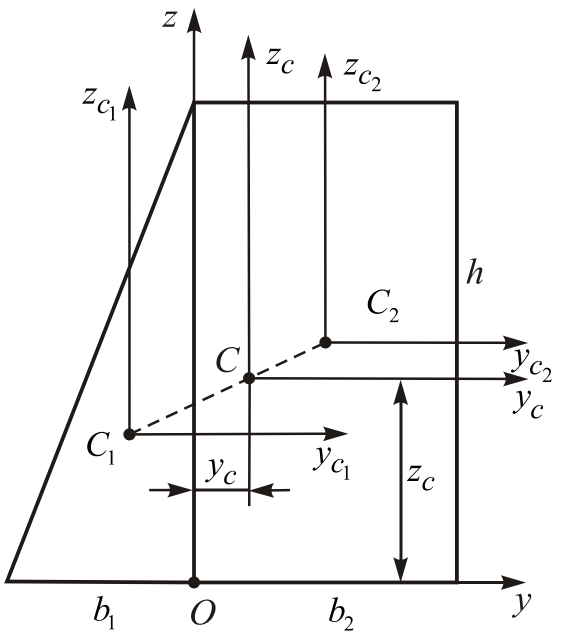


Fig. 37

Given: $b_1 = 30\,\text{mm}$, $b_2 = 10\,\text{mm}$, $h = 40\,\text{mm}$.

R.D.: x_c, y_c coordinates.

Solution Divide the area into two simplest figures: the **right triangle** and the **rectangle**, for which the centroids are know. Select an arbitrary reference system of axes y and z and determine the coordinates of the centroid using equation (10).

Substituting the numerical values into the foregoing expression we receive:

$$y_c = \frac{S_z}{A} = \frac{S_z^{\Delta} + S_z^{\square}}{A^{\Delta} + A^{\square}} = \frac{\sum_1^2 A_i y_{c_i}}{\sum_1^2 A_i} = \frac{\frac{b_1 h}{2}\left(-\frac{b_1}{3}\right) + b_2 h\left(+\frac{b_2}{2}\right)}{\frac{b_1 h}{2} + b_2 h} = \ldots,$$

$$z_c = \frac{S_y}{A} = \frac{S_y^{\Delta} + S_y^{\square}}{A^{\Delta} + A^{\square}} = \frac{\sum_1^2 A_i z_{c_i}}{\sum_1^2 A_i} = \frac{\frac{b_1 h}{2}\left(+\frac{h}{3}\right) + b_2 h\left(+\frac{h}{2}\right)}{\frac{b_1 h}{2} + b_2 h} = \ldots.$$

Centroids of simple figures

Circular sector

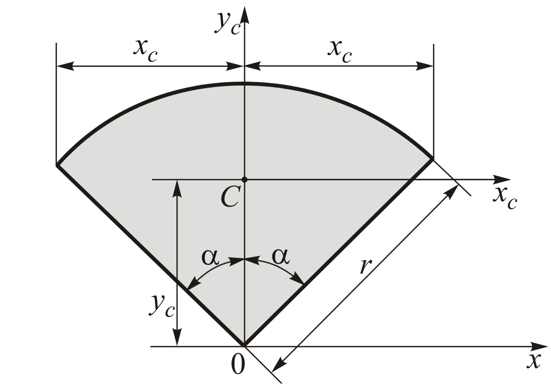


Fig. 38

Origin of axes at center of circle:
α =angle in radians $(\alpha \le \pi/2)$,

$$A = \alpha r^2,$$
$$x_C = r \sin\alpha,$$
$$y_C = \frac{2r\sin\alpha}{3\alpha}.$$

Circular segment

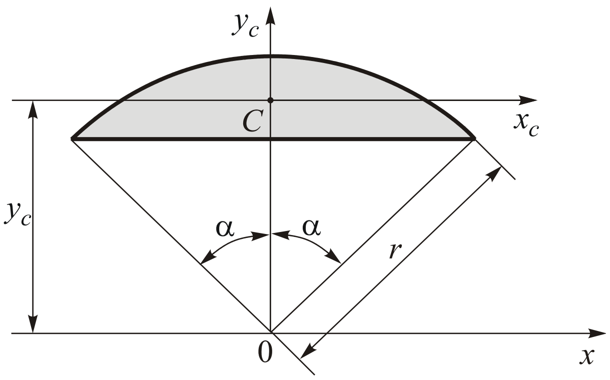


Fig 39

Origin of axes at center of circle:
α =angle in radians $(\alpha \le \pi/2)$,

$$A = r^2(\alpha - \sin\alpha\cos\alpha),$$
$$y_C = \frac{2r}{3}\left(\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}\right).$$

Isosceles triangle

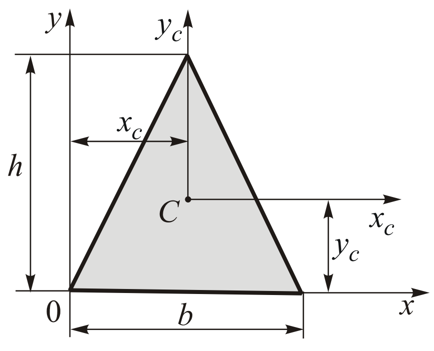


Fig 40

Origin of axes at centroid:

$$A = \frac{bh}{2},$$
$$x_C = \frac{b}{2},$$
$$y_C = \frac{h}{3}.$$

Parabolic semisegment

A parabolic semisegment *OAB* is bounded by the x axis, the y axis, and a parabolic curve having its vertex at *A* (Fig. 41). The equation of the curve is

$$y = f(x) = h\left(1 - \frac{x^2}{b^2}\right), \qquad (1)$$

in which b is the base and h is the height of the semisegment. Locate the centroid C of the semisegment.

To determine the coordinates x_c and y_c of the centroid C (Fig. 41), we will use equations:

$$x_c = \frac{S_y}{A}, \qquad y_c = \frac{S_x}{A}.$$

We begin by selecting an element of area dA in the form of a thin vertical strip of width dx and height y. The area of this differential element is

$$dA = ydx = h\left(1 - \frac{x^2}{b^2}\right)dx. \qquad (2)$$

Therefore, the area of the parabolic semisegment is

$$A = \int\limits_{(A)} dA = \int_0^b h\left(1 - \frac{x^2}{b^2}\right)dx = \frac{2bh}{3}. \qquad (3)$$

Note: This area is 2/3 of the area of the surrounding rectangle.

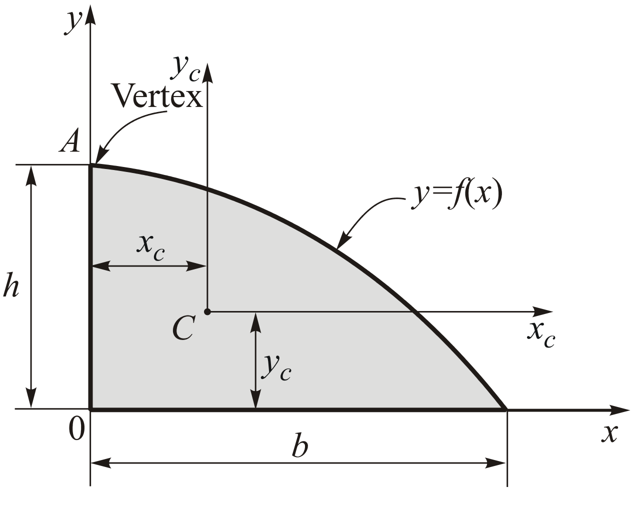


Fig. 41

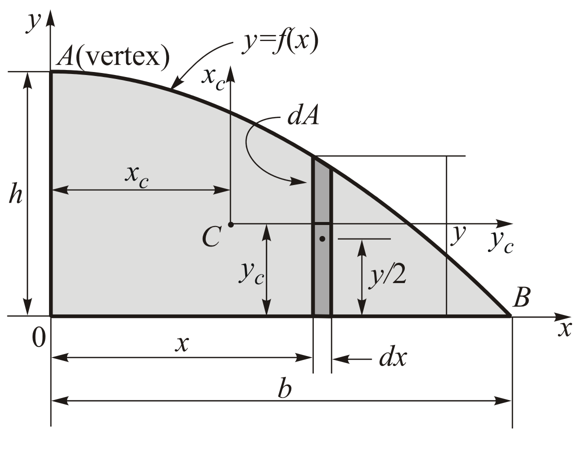


Fig. 42

The first moment of an element of area dA with respect to an axis is obtained by multiplying the area of the element by the distance from its centroid to the axis. Since the x and y coordinates of the centroid of the element shown in Fig. 42 are x and $y/2$, respectively, the first moments of the element with respect to the x and y axes are

$$S_x = \int \frac{y}{2} dA = \int_0^b \frac{h^2}{2}\left(1 - \frac{x^2}{b^2}\right)^2 dx = \frac{4bh^2}{15}, \tag{4}$$

$$S_y = \int x dA = \int_0^b hx\left(1 - \frac{x^2}{b^2}\right) dx = \frac{b^2 h}{4}, \tag{5}$$

in which we have substituted for dA from Eq. (2).

We can now determine the coordinates of the centroid C:

$$x_c = \frac{S_y}{A} = \frac{3b}{8}, \tag{6}$$

$$y_c = \frac{S_x}{A} = \frac{2h}{5}. \tag{7}$$

Notes: The centroid C of the parabolic semisegment may also be located by taking the element of area dA as a horizontal strip of height dy and width

$$x = b\sqrt{1 - \frac{y}{h}}. \tag{8}$$

This expression is obtained by solving Eq. (1) for x in terms of y.

Another possibility is to take the differential element as a rectangle of width dx and height dy. Then the expressions for A, S_x, and S_y are in the form of double integrals instead of single integrals.

Parabolic spandrel

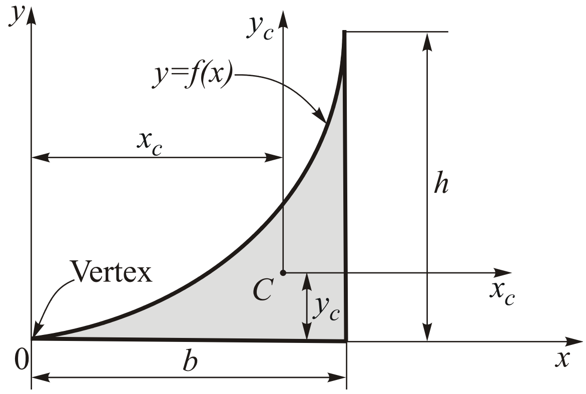


Fig 43

Origin of axes at vertex O:

$$y = f(x) = \frac{hx^2}{b^2},$$

$$A = \frac{bh}{3},$$

$$x_c = \frac{3b}{4},$$

$$y_c = \frac{3h}{10}.$$

Quarter circle

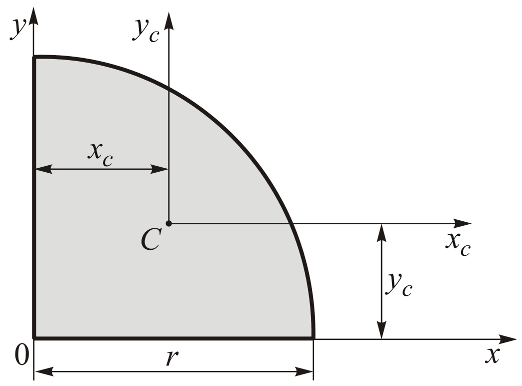


Fig 44

Origin of axes at center of circle O:

$$A = \frac{\pi r^2}{4},$$

$$x_c = y_c = \frac{4r}{3\pi}.$$

Quarter-circular spandrel

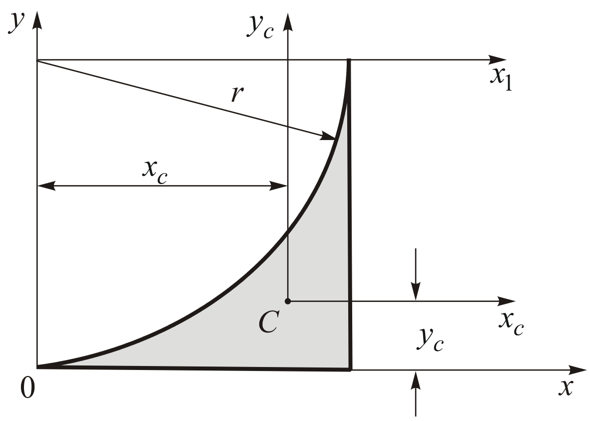


Fig 45

Origin of axes at point of tangency:

$$A = \left(1 - \frac{\pi}{4}\right) r^2,$$

$$x_c = \frac{2r}{3(4-\pi)} \approx 0.7766\, r,$$

$$y_c = \frac{(10-3\pi)r}{3(4-\pi)} \approx 0.2234\, r.$$

Rectangle

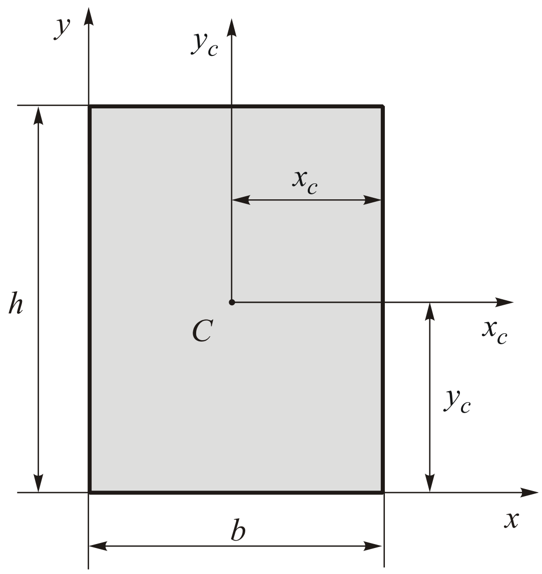


Fig 46

Origin of axes at centroid:

$$A = bh,$$

$$x_c = \frac{b}{2},$$

$$y_c = \frac{h}{2}.$$

Right triangle

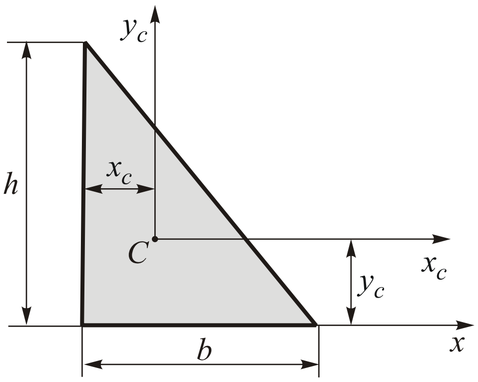


Fig 47

Origin of axes at centroid:

$$A = \frac{bh}{2},$$

$$x_C = \frac{b}{3},$$

$$y_C = \frac{h}{3}.$$

Semicircle

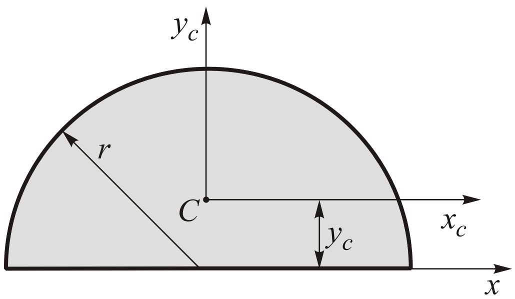


Fig 48

Origin of axes at centroid:

$$A = \frac{\pi r^2}{2},$$

$$y_C = \frac{4r}{3\pi}.$$

Sine wave

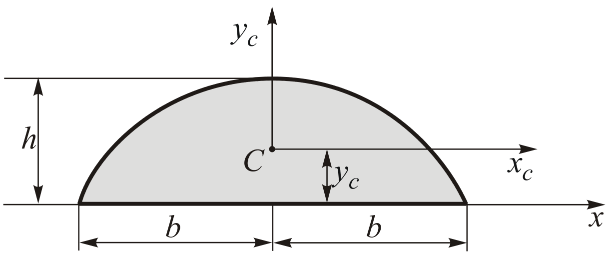


Fig 49

Origin of axes at centroid:

$$A = \frac{4bh}{\pi},$$

$$y_C = \frac{\pi h}{8}.$$

Thin circular arc

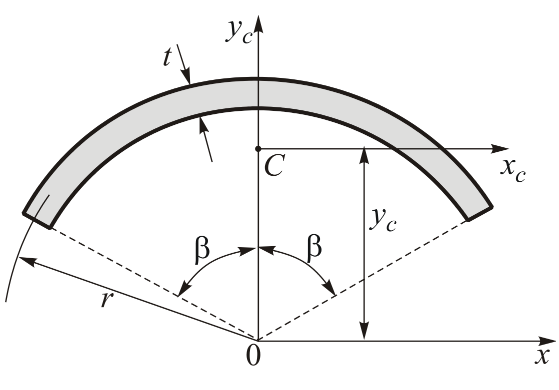


Fig 50

Origin of axes at center of circle. Approximate formulas for case when t is small:

β – angle in radians, $(\beta \le \pi/2)$;

$$A = 2\beta r t,$$

$$y_C = \frac{r \sin\beta}{\beta}.$$

Centroid of a trapezoid

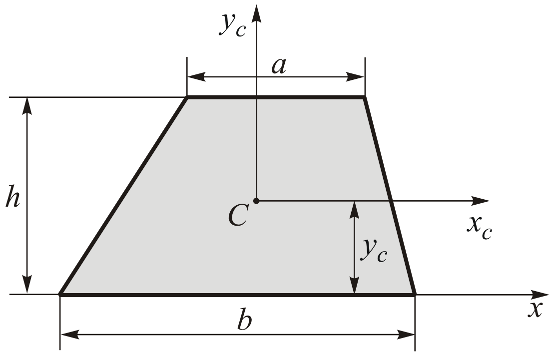


Fig 51

Origin of axes at centroid:

$$A = \frac{h(a+b)}{2},$$

$$y_c = \frac{h(2a+b)}{3(a+b)}.$$

Centroid of a triangle

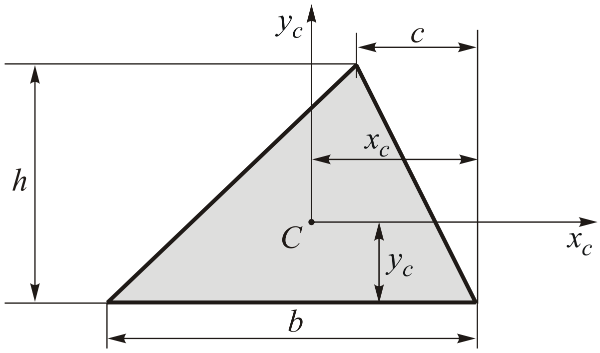


Fig 52

Origin of axes at centroid:

$$A = \frac{bh}{2},$$

$$x_c = \frac{b+c}{3},$$

$$y_c = \frac{h}{3}.$$

2.4 Axial Moments (Second Moments) and Product of Inertia

Take a cross section of a rod. Relate it to a system of co-ordinates *y*, *z*. Isolate an element *dA* from the area *A* with co-ordinates *y*, *z*. In addition to the static moment consider the following four integrals:

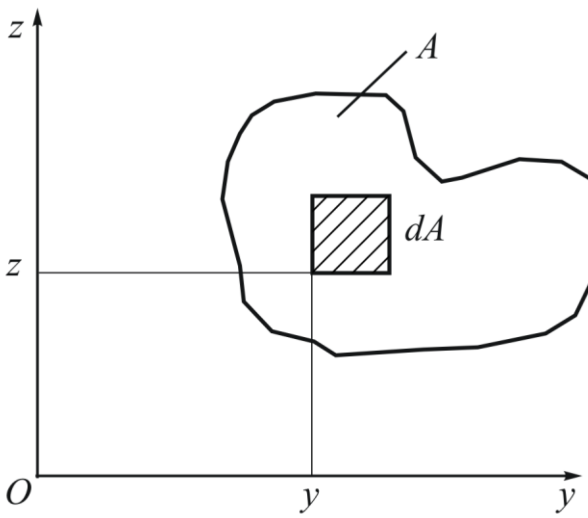


Fig. 53

$$I_y = \int_A z^2 dA, \quad I_z = \int_A y^2 dA, \tag{11}$$

$$I_{yz} = \int_A yz dA, \tag{12}$$

$$I_\rho = \int_A \rho^2 dA \,, \tag{13}$$

where the first two integrals (11) are called the **axial moments of inertia** of the section with respect to the *y* and *z* axes respectively.

The third integral (12) is called **the product of inertia** of the section with respect to two mutually perpendicular axes y and z.

The fourth integral (13) is called the **polar moment of inertia** of the section. The dimension of the moments of inertia is m^4 (meters in a power of four).

The axial and polar moments of inertia are always positive and cannot be equal to zero. ***The product of inertia may be positive, negative or equal to zero*** depending on the position of the axes. For example, this value with respect to any pair of axes is zero when either of the axes is an axis of symmetry.

Example 5 The calculation of the axial moment of inertia of a rectangle with respect to the central axes y_c **and** z_c (Fig. 54).

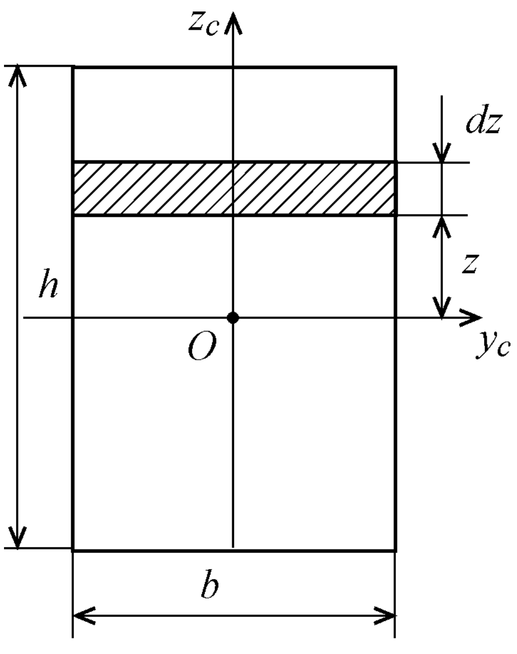


Fig. 54

Given: b and h – base and height of the rectangle respectively.

R.D.: central axial moments of inertia of a rectangle.

Solution Let us isolate an elementary area dA with the base b and the height dz at the distance z from the axis.

Since $dA=bdz$, then

$$I_y = \int_A z^2 dA = \int_{-\frac{h}{2}}^{+\frac{h}{2}} z^2 b dz = \frac{bh^3}{12}.$$

The moment of inertia with respect to the z-axis is found by a similar way:

$$I_z = \int_A y^2 dA = \int_{-\frac{b}{2}}^{+\frac{b}{2}} y^2 h dy = \frac{hb^3}{12}.$$

Example 6 Calculation of the central moment of inertia of a circular shape (Fig. 55).

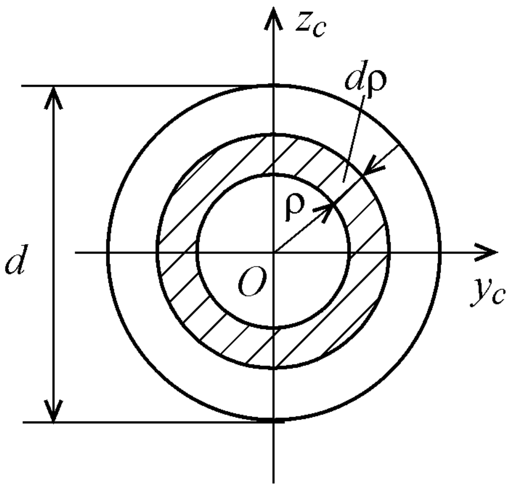


Fig. 55

Given: d – diameter of the circle.

R.D.: central axial moments of inertia.

Solution We take dA as $2\pi\rho d\rho$. Thus

$$I_\rho = \int_A \rho^2 dA = \int_0^{\frac{d}{2}} 2\pi\rho^3 d\rho = \frac{\pi d^4}{32}.$$

Referring to Fig 53, we find $\rho^2 = y^2 + z^2$.

That is

$I_\rho = \int_A \rho^2 dA = \int_A \left(y^2 + z^2\right) dA = I_y + I_z$. Using the symmetry, we can write

$$I_y = I_z = \frac{\pi d^4}{64}.$$

Example 7 Calculation of axial moments and product of inertia for right triangle relative to axes coincident with triangle legs (Fig. 56).

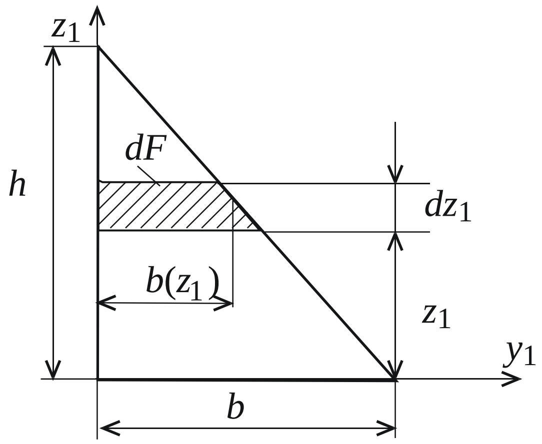


Fig. 56

Given: b, h

R.D.: I_{y_1}, I_{z_1}, $I_{y_1 z_1}$

Solution

(a) Calculation of axial moments of inertia

As preliminary determined

$$I_{y_1} = \int_F z_1^2 dA, \text{ where } dA_1 = b(z_1)dz_1.$$

Using similarity condition

$$\frac{b(z_1)}{b} = \frac{h - z_1}{h} \rightarrow b(z_1) = b\left(1 - \frac{z_1}{h}\right).$$

After substitution

$$I_{y_1} = \int_0^h b\left(1 - \frac{z_1}{h}\right) z_1^2 dz = b\left(\frac{z_1^3}{3} - \frac{z_1^4}{4h}\right)\Bigg|_0^h = \frac{bh^3}{12}.$$

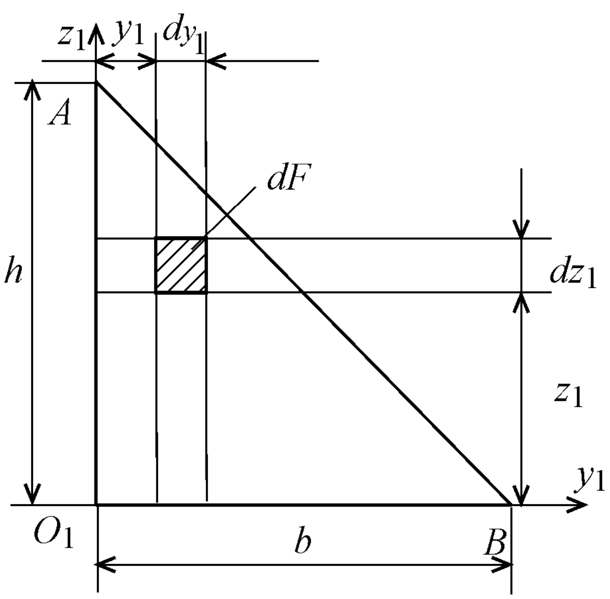


Fig. 57

Thus, $I_{y_1}^{\triangle} = \dfrac{bh^3}{12}$, by analogy $I_{z_1} = \dfrac{hb^3}{12}$.

(b) Calculation of product of inertia

It is known that for product of inertia

$$I_{y_1 z_1} = \int_F y_1 z_1 dA, \tag{a}$$

where $dA = by_1 dz_1$. (b)

Equation of inclined boundary *AB* is

$$\frac{z_1}{h} + \frac{y_1}{b} = 1, \text{ where}$$

$$z_1 = h\left(1 - \frac{y_1}{b}\right) \quad \text{or} \quad y_1 = b\left(1 - \frac{z_1}{h}\right). \tag{c}$$

After this, equation (a) may be rewritten:

$$I_{y_1 z_1} = \int_0^h \left(\int_0^{b\left(1-\frac{z_1}{h}\right)} y_1 z_1 dy_1 \right) dz_1 = \int_0^h z_1 \left[\left(\frac{y_1^2}{2} \right) \Bigg|_0^{b\left(1-\frac{z_1}{h}\right)} \right] dz_1 = \frac{b^2}{2} \int_0^h z_1 \left(1 - \frac{z_1}{h}\right)^2 dz_1 =$$

$$= \frac{b^2}{2} \left(\frac{z_1^2}{2} - \frac{2}{h}\frac{z_1^3}{3} + \frac{z_1^4}{4h^2} \right) \Bigg|_0^h = \frac{b^2 h^2}{24}.$$

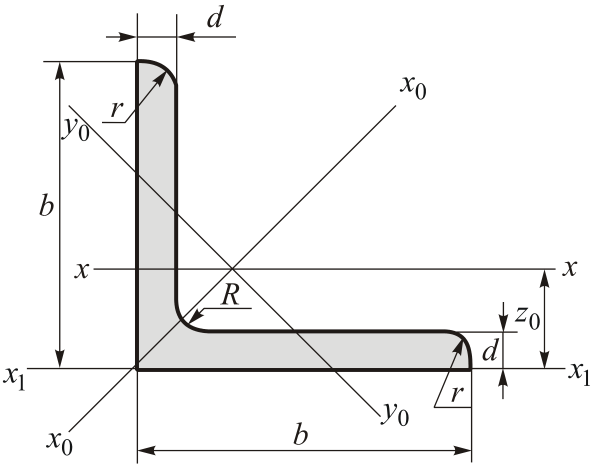
In result

$$I_{y_1 z_1}^{\triangle} = + \frac{b^2 h^2}{24}. \tag{d}$$

Note that the properties of structural elements such as **channels**, **angles** or ***I*-beams** are given in the tables of standard section (**assortments**). For some geometric figures central moments of inertia are presented below.

Assortments of steel products

Geometrical properties of angle sections with equal legs (L shapes) (GOST 8509-72)



b – width of web,

d – thickness,

I – moment of inertia,

i – radius of gyration,

z_0 – distance to centroid.

Fig. 58

Designation (number)	b	d	Area, cm²	Axes						z_0, cm	Mass per meter, kg
	mm			X – X		$X_0 - X_0$		$Y_0 - Y_0$			
				I_x, cm⁴	i_x, cm	$I_{x_0\ max}$, cm⁴	$i_{x_0\ max}$, cm	$I_{y_0\ min}$, cm⁴	$i_{y_0\ min}$, cm		
1	2	3	4	5	6	7	8	9	10	11	12
2	20	3	1,13	0,40	0,59	0,63	0,75	0,17	0,39	0,60	0,89
		4	1,46	0,50	0,58	0,78	0,73	0,22	0,38	0,64	1,15
2,5	25	3	1,43	0,81	0,75	1,29	0,95	0,34	0,49	0,73	1,12
		4	1,86	1,03	0,74	1,62	0,93	0,44	0,48	0,76	1,46
2,8	28	3	1,62	1,16	0,85	1,84	1,07	0,48	0,55	0,80	1,27
3,2	32	3	1,86	1,77	0,97	2,80	1,23	0,74	0,63	0,89	1,46
		4	2,43	2,26	0,96	3,58	1,21	0,94	0,62	0,94	1,91
3,6	36	3	2,10	2,56	1,10	4,06	1,39	1,06	0,71	0,99	1,65
		4	2,75	3,29	1,09	5,21	1,38	1,36	0,70	1,04	2,16
4	40	3	2,35	3,55	1,23	5,63	1,55	1,47	0,79	1,09	1,85
		4	3,08	4,58	1,22	7,26	1,53	1,90	0,78	1,13	2,42
		5	3,79	5,53	1,20	8,75	1,54	2,30	0,79	1,17	2,97
4,5	45	3	2,65	5,13	1,39	8,13	1,75	2,12	0,89	1,21	2,08
		4	3,48	6,63	1,38	10,50	1,74	2,74	0,89	1,26	2,73
		5	4,29	8,03	1,37	12,70	1,72	3,33	0,88	1,30	3,37
5	50	3	2,96	7,11	1,55	11,30	1,95	2,95	1,00	1,33	2,32
		4	3,89	9,21	1,54	14,60	1,94	3,80	0,99	1,38	3,05
		5	4,80	11,20	1,53	17,80	1,92	4,63	0,98	1,42	3,77
5,6	56	4	4,38	13,10	1,73	20,80	2,18	5,41	1,11	1,52	3,44
		5	5,41	16,00	1,72	25,40	2,16	6,59	1,10	1,57	4,25
6,3	63	4	4,96	18,90	1,95	29,90	2,45	7,81	1,25	1,69	3,90
		5	6,13	23,10	1,94	36,60	2,44	9,52	1,25	1,74	4,81
		6	7,28	27,10	1,93	42,90	2,43	11,20	1,24	1,78	5,72
7	70	4,5	6,20	29,0	2,16	46,0	2,72	12,0	1,39	1,88	4,87
		5	6,86	31,9	2,16	50,7	2,72	13,2	1,39	1,90	5,38
		6	8,15	37,6	2,15	59,6	2,71	15,5	1,38	1,94	6,39
		7	9,42	43,0	2,14	68,2	2,69	17,8	1,37	1,99	7,39
		8	10,70	48,2	2,13	76,4	2,68	20,0	1,37	2,02	8,37

(finished)

1	2	3	4	5	6	7	8	9	10	11	12
7,5	75	5	7,39	39,5	2,31	62,6	2,91	16,4	1,49	2,02	5,80
		6	8,78	46,6	2,30	73,9	2,90	19,3	1,48	2,06	6,89
		7	10,10	53,3	2,29	84,6	2,89	22,1	1,48	2,10	7,96
		8	11,50	59,8	2,28	94,6	2,87	24,8	1,47	2,15	9,02
		9	12,80	66,1	2,27	105,0	2,86	27,5	1,46	2,18	10,10
8	80	5,5	8,63	52,7	2,47	83,6	3,11	21,8	1,59	2,17	6,78
		6	9,38	57,0	2,47	90,4	3,11	23,5	1,58	2,19	7,36
		7	10,80	65,3	2,45	104,0	3,09	27,0	1,58	2,23	8,51
		8	12,30	73,4	2,44	116,0	3,08	30,3	1,57	2,27	9,65
9	90	6	10,60	82,1	2,78	130,0	3,50	34,0	1,79	2,43	8,33
		7	12,30	94,3	2,77	150,0	3,49	38,9	1,78	2,47	9,64
		8	13,90	106,0	2,76	168,0	3,48	43,8	1,77	2,51	10,90
		9	15,60	118,0	2,75	186,0	3,46	48,6	1,77	2,55	12,20
10	100	6,5	12,80	122,0	3,09	193,0	3,88	50,7	1,99	2,68	10,10
		7	13,80	131,0	3,08	207,0	3,88	54,2	1,98	2,71	10,80
		8	15,60	147,0	3,07	233,0	3,87	60,9	1,98	2,75	12,20
		10	19,20	179,0	3,05	284,0	3,84	74,1	1,96	2,83	15,10
		12	22,80	209,0	3,03	331,0	3,81	86,9	1,95	2,91	17,90
		14	26,30	237,0	3,00	375,0	3,78	99,3	1,94	2,99	20,60
		16	29,70	264,0	2,98	416,0	3,74	112,0	1,94	3,06	23,30
11	110	7	15,20	176,0	3,40	279,0	4,29	72,7	2,19	2,96	11,90
		8	17,20	198,0	3,39	315,0	4,28	81,8	2,18	3,00	13,50
12,5	125	8	19,7	294	3,87	467	4,87	122	2,49	3,36	15,5
		9	22,0	327	3,86	520	4,86	135	2,48	3,40	17,3
		10	24,3	360	3,85	571	4,84	149	2,47	3,45	19,1
		12	28,9	422	3,82	670	4,82	174	2,46	3,53	22,7
		14	33,4	482	3,80	764	4,78	200	2,45	3,61	26,2
		16	37,8	539	3,78	853	4,75	224	2,44	3,68	29,6
14	140	9	24,7	466	4,34	739	5,47	192	2,79	3,78	19,4
		10	27,3	512	4,33	814	5,46	211	2,78	3,82	21,5
		12	32,5	602	4,31	957	5,43	248	2,76	3,90	25,5
16	160	10	31,4	774	4,96	1229	6,25	319	3,19	4,30	24,7
		11	34,4	844	4,95	1341	6,24	348	3,18	4,35	27,0
		12	37,4	913	4,94	1450	6,23	376	3,17	4,39	29,4
		14	43,3	1046	4,92	1662	6,20	431	3,16	4,47	34,0
		16	49,1	1175	4,89	1866	6,17	485	3,14	4,55	38,5
		18	54,8	1299	4,87	2061	6,13	537	3,13	4,63	43,0
		20	60,4	1419	4,85	2248	6,10	589	3,12	4,70	47,4
18	180	11	38,8	1216	5,60	1133	7,06	500	3,59	4,85	30,5
		12	42,2	1317	5,59	2093	7,04	540	3,58	4,89	33,1
20	200	12	47,1	1823	6,22	2896	7,84	749	3,99	5,37	37,0
		13	50,9	1961	6,21	3116	7,83	805	3,98	5,42	39,9
		14	54,6	2097	6,20	3333	7,81	861	3,97	5,46	42,8
		16	62,0	2363	6,17	3755	7,78	970	3,96	5,54	48,7
		20	76,5	2871	6,12	4560	7,72	1182	3,93	5,70	60,1
		25	94,3	3466	6,06	5494	7,63	1438	3,91	5,89	74,0
		30	111,5	4020	6,00	6351	7,55	1688	3,89	6,07	87,6
22	220	14	60,4	2814	6,83	1170	8,60	1159	4,38	5,93	47,4
		16	68,6	3175	6,81	5045	8,58	1306	4,36	6,02	53,8
25	250	16	78,4	4717	7,76	7492	9,78	1942	4,98	6,75	61,5
		18	87,7	5247	7,73	8337	9,75	2158	4,96	6,83	68,9
		20	97,0	5765	7,71	9160	9,72	2370	4,94	6,91	76,1
		22	106,1	6270	7,69	9961	9,69	2579	4,93	7,00	83,3
		25	119,7	7006	7,65	11125	9,64	2887	4,91	7,11	94,0
		28	133,1	7717	7,61	12244	9,59	3190	4,89	7,23	104,5
		30	142,0	8177	7,59	12965	9,56	3389	4,89	7,31	111,4

Geometrical properties of angle sections with unequal legs (L shapes) (GOST 8510-72)

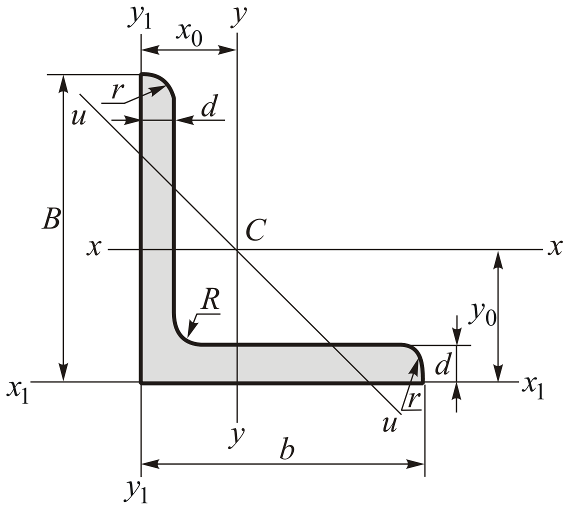


Fig. 59

B – width of larger leg,

b – width of smaller leg,

d – thickness of legs,

I – moment of inertia,

i – radius of gyration,

x_0, y_0 – distances from the centroid to the back of the legs.

Designation (number)	B	b	d	Area, cm^2	Axes								tan α	Mass per meter, kg
	mm				$X-X$		$Y-Y$		—		x_0	y_0		
					I_x, cm^4	i_x, cm	I_y, cm^4	i_y, cm	$I_{u\,min}$, cm^4	$i_{u\,min}$, cm	cm	cm		
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2,5/1,6	25	16	3	1,16	0,70	0,78	0,22	0,44	0,13	0,34	0,42	0,86	0,392	0,91
3,2/2	32	20	3	1,49	1,52	1,01	0,46	0,55	0,28	0,43	0,49	1,08	0,382	1,17
			4	1,94	1,93	1,00	0,57	0,54	0,35	0,43	0,53	1,12	0,374	1,52
4/2,5	40	25	3	1,89	3,06	1,27	0,93	0,70	0,56	0,54	0,59	1,32	0,385	1,48
			4	2,47	3,93	1,26	1,18	0,69	0,71	0,54	0,63	1,37	0,381	1,94
4,5/2,8	45	28	3	2,14	4,41	1,43	1,32	0,79	0,79	0,61	0,64	1,47	0,382	1,68
			4	2,80	5,68	1,42	1,69	0,78	1,02	0,60	0,68	1,51	0,379	2,20
5/3,2	50	32	3	2,42	6,17	1,60	1,99	0,91	1,18	0,70	0,72	1,60	0,403	1,90
			4	3,17	7,98	1,59	2,56	0,90	1,52	0,69	0,76	1,85	0,401	1,49
5,6/3,6	56	36	4	3,58	11,40	1,78	3,70	1,02	2,19	0,78	0,84	1,82	0,406	2,81
			5	4,41	13,80	1,77	4,48	1,01	2,66	0,78	0,88	1,86	0,404	3,46
6,3/4,0	63	40	4	4,04	16,30	2,01	5,16	1,13	3,07	0,87	0,91	2,03	0,397	3,17
			5	4,98	19,90	2,00	6,26	1,12	3,72	0,86	0,95	2,08	0,396	3,91
			6	5,90	23,30	1,99	7,28	1,11	4,36	0,86	0,99	2,12	0,393	4,63
			8	7,68	29,60	1,96	9,15	1,09	5,58	0,85	1,07	2,20	0,386	6,03
7/4,5	70	45	5	5,59	27,80	2,23	9,05	1,27	5,34	0,98	1,05	2,28	0,406	4,39
7,5/5	75	50	5	6,11	34,80	2,39	12,50	1,43	7,24	1,09	1,17	2,39	0,436	4,79
			6	7,25	40,90	2,38	14,60	1,42	8,48	1,08	1,21	2,44	0,435	5,69
			8	9,47	52,40	2,35	18,50	1,40	10,90	1,07	1,29	2,52	0,430	7,43
8/5	80	50	5	6,36	41,60	2,56	12,70	1,41	7,58	1,09	1,13	2,60	0,387	4,99
			6	7,55	49,00	2,55	14,80	1,40	8,88	1,08	1,17	2,65	0,386	5,92
9/5,6	90	56	5,5	7,86	65,3	2,88	19,7	1,58	11,8	1,22	1,26	2,92	0,384	6,17
			6,0	8,54	70,6	2,88	21,2	1,58	12,7	1,22	1,28	2,95	0,384	6,70
			8,0	11,18	90,9	2,85	27,1	1,56	16,3	1,21	1,36	3,04	0,380	8,77
10/6,3	100	63	6,0	9,59	98,3	3,20	30,6	1,79	18,2	1,38	1,42	3,23	0,393	7,53
			7,0	11,10	113,0	3,19	35,0	1,78	20,8	1,37	1,46	3,28	0,392	8,70

(finished)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
			8,0	12,6	127,0	3,18	39,2	1,77	23,4	1,36	1,50	3,32	0,391	9,87
			10,0	15,50	154,0	3,15	47,1	1,75	28,3	1,35	1,58	3,40	0,387	2,10
11/7	110	70	6,5	11,40	142,0	3,53	45,6	2,00	26,9	1,53	1,58	3,55	0,402	9,98
			8,0	13,90	172,0	3,51	54,6	1,98	32,3	1,52	1,64	3,61	0,400	10,90
12,5/8	125	80	7,0	14,10	227,0	4,01	73,7	2,29	43,4	1,76	1,80	4,01	0,407	11,00
			8,0	16,00	256,0	4,00	83,0	2,28	48,8	1,75	1,84	4,05	0,406	12,50
			10,0	19,70	312,0	3,98	100,0	2,26	59,3	1,74	1,92	4,14	0,404	15,50
			12,0	23,40	365,0	3,95	117,0	2,24	69,5	1,72	2,00	4,22	0,400	18,30
14/9	140	90	8,0	18,00	364,0	4,49	120,0	2,58	70,3	1,98	2,03	4,49	0,411	14,10
			10,0	22,20	444,0	4,47	146,0	2,56	58,5	1,96	2,12	4,58	0,409	17,50
16/10	160	100	9,0	22,90	606,0	5,15	186,0	2,85	110,0	2,20	2,23	5,19	0,391	18,0
			10,0	25,30	667,0	5,13	204,0	2,84	121,0	2,19	2,28	5,23	0,390	19,80
			12,0	30,00	784,0	5,11	239,0	2,82	142,0	2,18	2,36	5,32	0,388	23,60
			14,0	34,70	897,0	5,09	272,0	2,80	162,0	2,16	2,43	5,40	0,385	27,30
18/11	180	110	10,0	28,30	952,0	5,80	276,0	3,12	165,0	2,42	2,44	5,88	0,375	22,20
			12,0	33,70	1123	5,77	324,0	3,10	194,0	2,40	2,52	5,97	0,374	26,40
20/12,5	200	125	11	34,9	1449	6,45	446	3,58	264	2,75	2,79	6,5	0,392	27,4
			12	37,9	1568	6,43	482	3,57	285	2,74	2,83	6,54	0,392	29,7
			14	43,9	1801	6,41	551	3,54	327	2,73	2,91	6,62	0,390	34,4
			16	49,8	2026	6,38	617	3,52	367	2,72	2,99	6,71	0,388	39,1
25/16	250	160	12	48,3	3147	8,07	1032	4,62	604	3,54	3,53	7,97	0,410	37,9
			16	63,6	4091	8,02	1333	4,58	781	3,50	3,69	8,14	0,408	49,9
			18	71,1	4545	7,99	1475	4,56	866	3,49	3,77	8,23	0,407	55,8
			20	78,5	4987	7,97	1613	4,53	949	3,48	3,85	8,31	0,405	67,7

Geometrical properties of channel sections (C shapes) (GOST 8240-72)

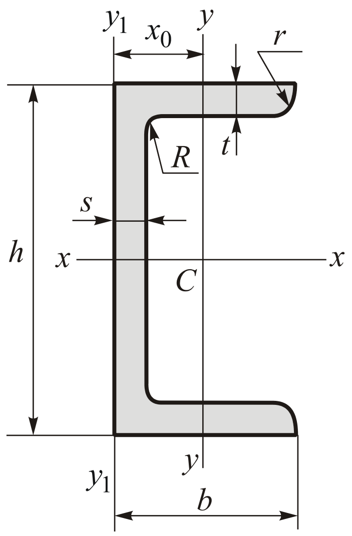


Fig. 60

h – height of a beam,

b – width of a flange,

s – thickness of a web,

t – average thickness of a flange,

W – sectional modulus,

i – radius of gyration,

S_x – first moment of area,

I – moment of inertia,

x_0 – distance from the centroid to the back of the web.

Designation (number)	Dimensions, mm				Area, cm^2	I_x, cm^4	W_x, cm^3	i_x, cm	S_x, cm^3	I_y, cm^4	W_y, cm^3	i_y, cm	x_0, cm	Weight per meter, kg
	h	b	s	t										
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	50	32	4,4	7,0	6,16	22,8	9,1	1,92	5,59	5,61	2,75	0,954	1,16	4,84
6,5	65	36	4,4	7,2	7,51	48,6	15,0	2,54	9,0	8,7	3,68	1,08	1,24	5,90
8	80	40	4,5	7,4	8,98	89,4	22,4	3,16	13,3	12,8	4,75	1,19	1,31	7,05
10	100	46	4,5	7,6	10,9	174	34,8	3,99	20,4	20,4	6,46	1,37	1,44	8,59
12	120	52	4,8	7,8	13,3	304	50,6	4,78	29,6	31,2	8,52	1,53	1,54	10,4

(finished)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
14	140	58	4,9	8,1	15,6	491	70,2	5,60	40,8	45,4	11,0	1,70	1,67	12,3
14a	140	62	4,9	8,7	17,0	545	77,8	5,66	45,1	57,5	13,3	1,84	1,87	13,3
16	160	64	5,0	8,4	18,1	747	93,4	6,42	54,1	63,3	13,8	1,87	1,80	14,2
16a	160	68	5,0	9,0	19,5	823	103	6,49	59,4	78,8	16,4	2,01	2,00	15,3
18	180	70	5,1	8,7	20,7	1090	121	7,24	69,8	86	17,0	2,04	1,94	16,3
18a	180	74	5,1	9,3	22,2	1190	132	7,32	76,1	105	20,0	2,18	2,13	17,4
20	200	76	5,2	9,0	23,4	1520	152	8,07	87,8	113	20,5	2,20	2,07	18,4
20a	200	80	5,2	9,7	25,2	1670	167	8,15	95,9	139	24,2	2,35	2,28	19,8
22	220	82	5,4	9,5	26,7	2110	192	8,89	110	151	25,1	2,37	2,21	21,0
22a	220	87	5,4	10,2	28,8	2330	212	8,99	121	187	30,0	2,55	2,46	22,6
24	240	90	5,6	10,0	30,6	2900	242	9,73	139	208	31,6	2,60	2,42	24,0
24a	240	95	5,6	10,7	32,9	3180	265	9,84	151	254	37,2	2,78	2,67	25,8
27	270	95	6,0	10,5	35,2	4160	308	10,9	178	262	37,3	2,73	2,47	27,7
30	300	100	6,5	11,0	40,5	5810	387	12,0	224	327	43,6	2,84	2,52	31,8
33	330	105	7,0	11,7	46,5	7980	484	13,1	281	410	51,8	2,97	2,59	36,5
36	360	110	7,5	12,6	53,4	10820	601	14,2	350	513	61,7	3,10	2,68	41,9
40	400	115	8,0	13,5	61,5	15220	761	15,7	444	642	73,4	3,26	2,75	48,3

Geometrical properties of S shapes (I-beam sections) (GOST 8239-72)

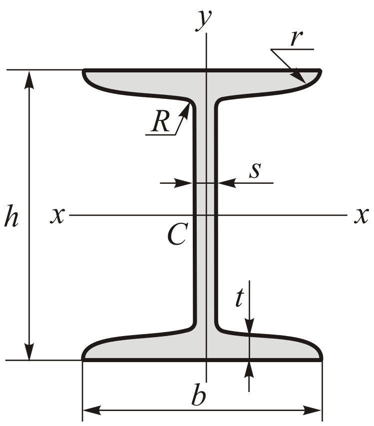


Fig. 61

h – height of a beam,

b – width of a flange,

s – thickness of a web,

t – average thickness of a flange,

I – axial moment of inertia,

W – sectional modulus,

i – radius of gyration,

S_x – first moment of a half-section.

Designation (number)	Dimensions, mm				Area, cm^2	I_x, cm^4	W_x, cm^3	i_x, cm	S_x, cm^3	I_y, cm^4	W_y, cm^3	i_y, cm	Mass per meter, kg
	h	b	s	t									
1	2	3	4	5	6	7	8	9	10	11	12	13	14
10	100	55	4,5	7,2	12,0	198	39,7	4,06	23,0	17,9	6,49	1,22	9,46
12	120	64	4,8	7,3	14,7	350	58,4	4,88	33,7	27,9	8,72	1,38	11,5
14	140	73	4,9	7,5	17,4	572	81,7	5,73	46,8	41,9	11,5	1,55	13,7
16	160	81	5,0	7,8	20,2	873	109	6,57	62,	58,6	14,5	1,70	15,9
18	180	90	5,1	8,1	23,4	1290	143	7,42	81,4	82,6	18,4	1,88	18,4
18a	180	100	5,1	8,3	25,4	1430	159	7,51	89,8	114	22,8	2,12	19,9
20	200	100	5,2	8,4	26,8	1840	184	8,28	104	115	23,1	2,07	21,0
20a	200	110	5,2	8,6	28,9	2030	203	8,37	114	155	28,2	2,32	22,7
22	220	110	5,4	5,4	30,6	2550	232	9,13	131	157	28,6	2,27	24,0
22a	220	120	5,4	8,9	32,8	2790	254	9,22	143	206	34,3	2,50	25,8
24	240	115	5,6	9,5	34,8	3460	289	9,97	163	198	34,5	2,37	27,3

(finished)

1	2	3	4	5	6	7	8	9	10	11	12	13	14
24a	240	125	5,6	9,8	37,5	3800	317	10,1	178	260	41,6	2,63	29,4
27	270	125	6,0	9,8	40,2	5010	371	11,2	210	260	41,5	2,54	31,5
27a	270	135	6,0	10,2	43,2	5500	407	11,3	229	337	50,0	2,80	33,9
30	300	135	6,5	10,2	46,5	7080	472	12,3	268	337	49,9	2,69	36,5
30a	300	145	6,5	10,7	49,9	7780	518	12,5	292	436	60,1	2,95	39,2
33	330	140	7,0	11,2	53,8	9840	597	13,5	389	419	69,9	2,79	42,2
36	360	145	7,5	12,3	61,9	13380	743	14,7	423	519	71,1	2,89	48,6
40	400	155	8,3	13,0	72,6	19062	953	16,2	545	667	86,1	3,03	57,0
45	450	160	9	14,2	84,7	27696	1231	18,1	708	808	101	3,09	66,5
50	500	170	10	15,2	100	39727	1589	19,9	919	1043	123	3,23	78,5
55	550	180	11	16,5	118	35962	2035	21,8	1181	1366	151	3,39	92,7
60	600	190	12	17,8	138	76806	2560	23,6	1481	1725	182	3,54	108

Centroidal axial moments of inertia for simple figures

Circle

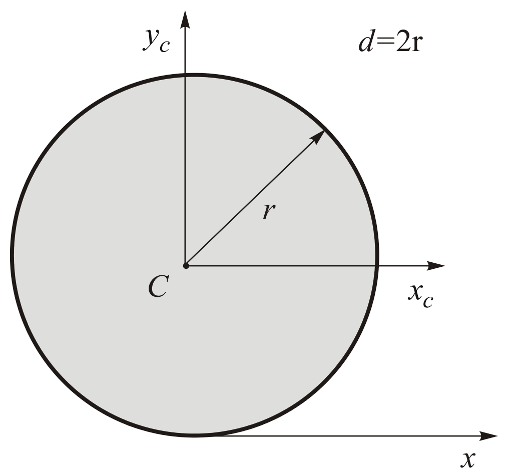


Fig. 62

Origin of axes at center of circle:

$$A=\pi r^2=\frac{\pi d}{4},$$

$$I_{x_c}=I_{y_c}=\frac{\pi r^4}{4}=\frac{\pi d^4}{64},$$

$$I_{xy}=0,\ I_p=\frac{\pi r^4}{2}=\frac{\pi d^4}{32},$$

$$I_x=\frac{5\pi r^4}{4}=\frac{5\pi d^4}{64}.$$

Circle with core removed

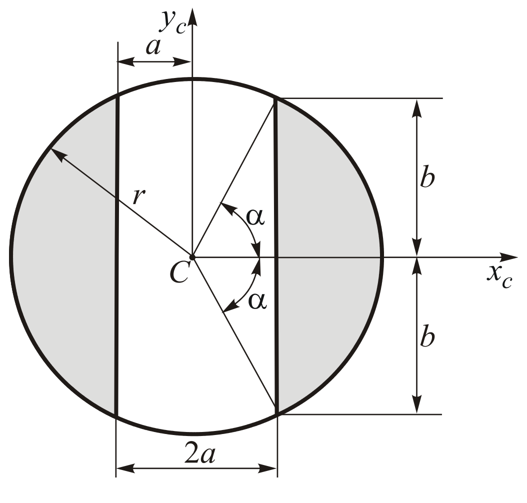


Fig. 63

Origin of axes at center of circle:

α =angle in radians, $(\alpha \le \pi/2)$;

$$\alpha=\arccos\frac{a}{r},\ \ b=\sqrt{r^2-a^2};$$

$$A=2r^2\left(\alpha-\frac{ab}{r^2}\right),$$

$$I_{x_c}=\frac{r^4}{6}\left(3\alpha-\frac{3ab}{r^2}-\frac{2ab^3}{r^4}\right),$$

$$I_{x_c}=\frac{r^4}{2}\left(\alpha-\frac{ab}{r^2}-\frac{2ab^3}{r^4}\right),\ \ I_{x_c y_c}=0.$$

Circular sector

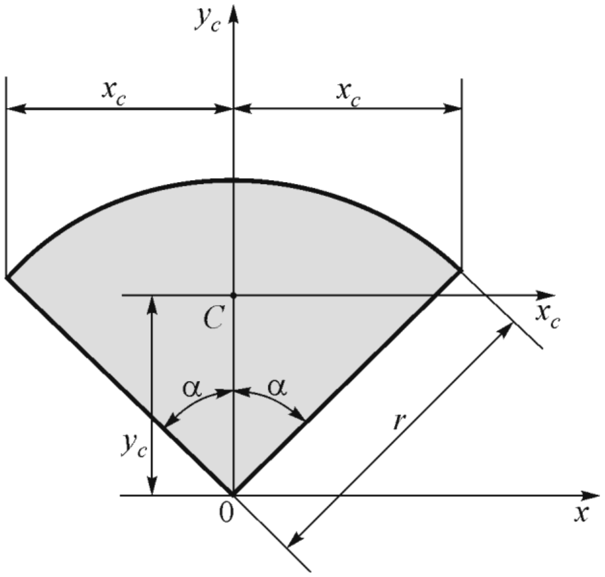


Fig. 64

Origin of axes at center of circle:
α =angle in radians, $(\alpha \le \pi/2)$;

$$A = \alpha r^2, \quad x_c = r\sin\alpha, \quad y_c = \frac{2r\sin\alpha}{3\alpha};$$

$$I_{x_c} = \frac{r^4}{4}(\alpha + \sin\alpha\cos\alpha),$$

$$I_{y_c} = \frac{r^4}{4}(\alpha - \sin\alpha\cos\alpha),$$

$$I_{x_c y_c} = I_{xy_c} = 0,$$

$$I_\rho = \frac{\alpha r^4}{2}.$$

Circular segment

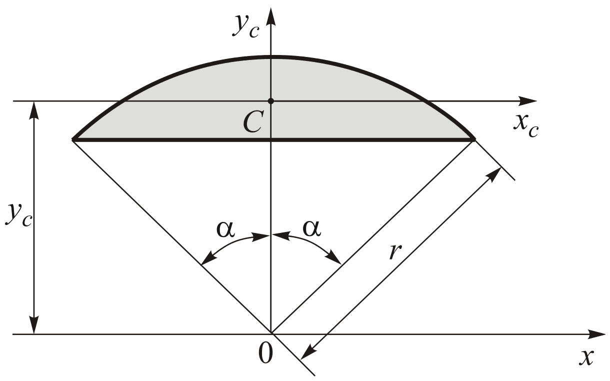


Fig. 65

Origin of axes at center of circle:
α =angle in radians, $(\alpha \le \pi/2)$;

$$y_c = \frac{2r}{3}\left(\frac{\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha}\right),$$

$$I_x = \frac{r^4}{4}(\alpha - \sin\alpha\cos\alpha + 2\sin^3\alpha),$$

$$I_{x_c y_c} = I_{xy_c} = 0,$$

$$I_{y_c} = \frac{r^4}{12}(3\alpha - 3\sin\alpha\cos\alpha - 2\sin^3\alpha\cos\alpha)$$

Ellipse

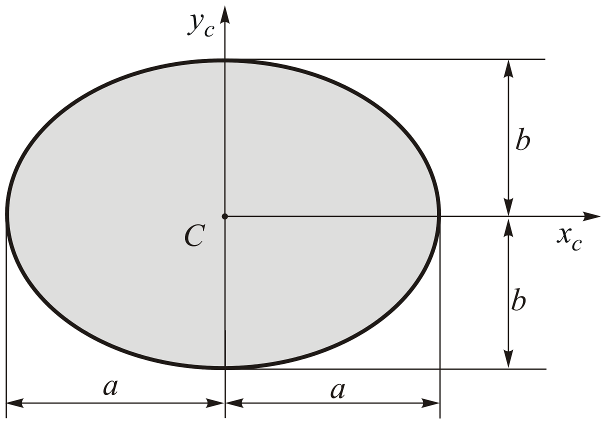


Fig. 66

Origin of axes at centroid:

$$A = \pi ab,$$

$$I_{x_c} = \frac{\pi ab^3}{4}, \quad I_{y_c} = \frac{\pi ba^3}{4};$$

$$I_{x_c y_c} = 0,$$

$$I_p = \frac{\pi ab}{4}(b^2 + a^2).$$

$$\text{Circumference} \approx \pi[1.5(a+b) - \sqrt{ab}],$$

$$(a/3 \le b \le a),$$

$$\approx 4.17b^2/a + 4a, \quad (0 \le b \le a/3).$$

Isosceles triangle

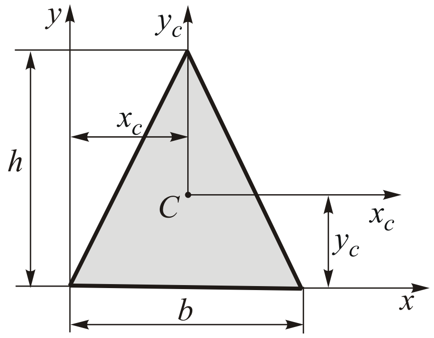


Fig. 67

Origin of axes at centroid:

$$A=\frac{bh}{2},\quad x_c=\frac{b}{2},\quad y_c=\frac{h}{3};$$

$$I_{x_c}=\frac{bh^3}{36},\quad I_{y_c}=\frac{hb^3}{48},\quad I_{x_c y_c}=0;$$

$$I_\rho=\frac{bh}{144}(4h^2+3b^2),\quad I_x=\frac{bh^3}{12}.$$

Note: For an equilateral triangle,

$$h=\sqrt{3}b/2.$$

Parabolic semisegment

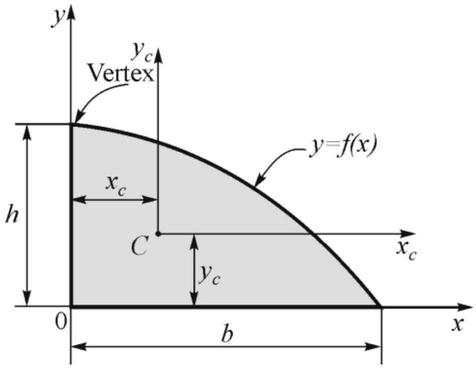


Fig. 68

Origin of axes at corner:

$$y=f(x)=h\left(1-\frac{x^2}{b^2}\right),$$

$$A=\frac{2bh}{3},\quad x_c=\frac{3b}{8},\quad y_c=\frac{2h}{5};$$

$$I_x=\frac{16bh^3}{105},\quad I_y=\frac{2hb^3}{15},\quad I_{xy}=\frac{b^2h^2}{12}.$$

Parabolic spandrel

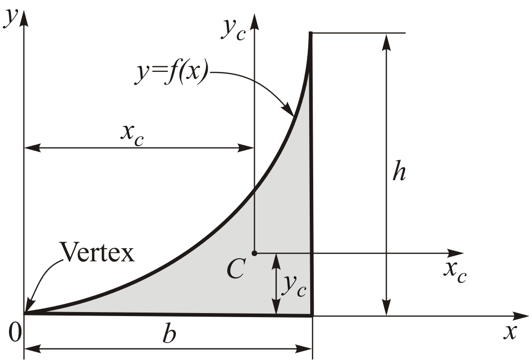


Fig. 69

Origin of axes at vertex:

$$y=f(x)=\frac{hx^3}{b^2},$$

$$A=\frac{bh}{3},\quad x_c=\frac{3b}{4},\quad y_c=\frac{3h}{10};$$

$$I_x=\frac{bh^3}{21},\quad I_y=\frac{hb^3}{5},\quad I_{xy}=\frac{b^2h^2}{12}.$$

Quarter circle

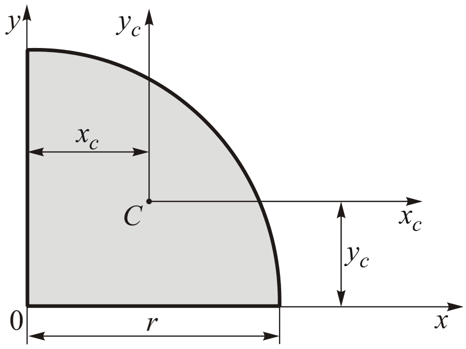


Fig. 70

Origin of axes at center of circle:

$$A=\frac{\pi r^2}{4},\quad x_c=y_c=\frac{4r}{3\pi};$$

$$I_x=I_y=\frac{\pi r^4}{16},\quad I_{xy}=\frac{r^4}{8};$$

$$I_{x_c}=I_{y_c}=\frac{(9\pi^2-64)r^4}{144\pi}\approx 0.05488r^4.$$

Quarter-circular spandrel

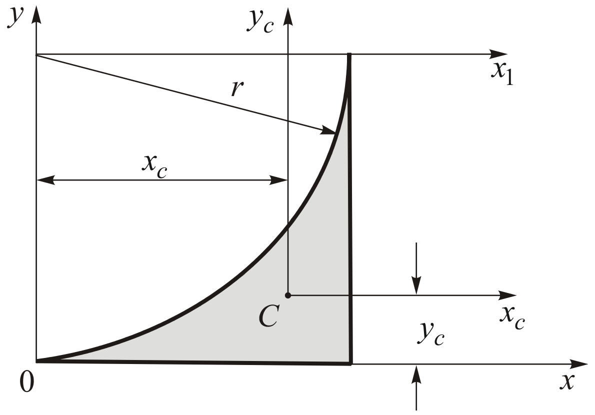


Fig. 71

Origin of axes at point of tangency:

$$A = \left(1 - \frac{\pi}{4}\right) r^2,$$

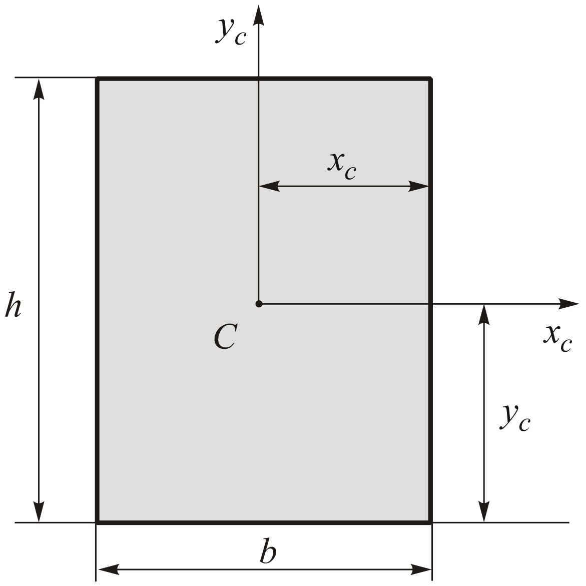
$$x_c = \frac{2r}{3(4-\pi)} \approx 0.7766r,$$

$$y_c = \frac{(10-3\pi)r}{3(4-\pi)} \approx 0.2234r,$$

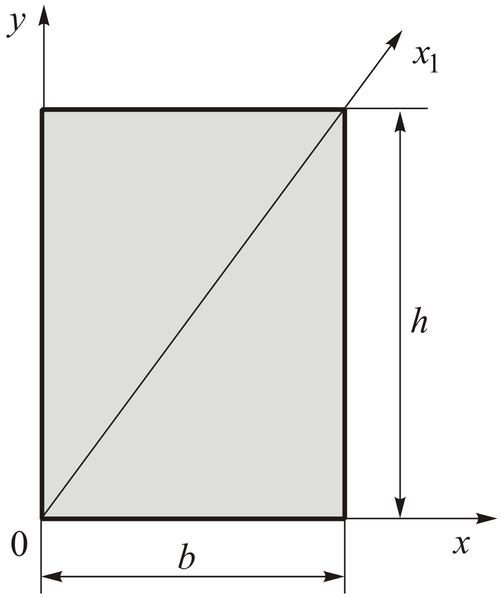
$$I_x = \left(1 - \frac{5\pi}{16}\right) r^4 \approx 0.01825 r^4,$$

$$I_y = I_{x_1} = \left(\frac{1}{3} - \frac{\pi}{16}\right) r^4 \approx 0.1370 r^4.$$

Rectangle



a



b

Fig 72

a) Origin of axes at centroid:

$$A = bh, \quad x_c = \frac{b}{2}, \quad y_c = \frac{h}{2};$$

$$I_{x_c} = \frac{bh^3}{12}, \quad I_{y_c} = \frac{hb^3}{12}, \quad I_{x_c y_c} = 0;$$

$$I_\rho = \frac{bh}{12}(h^2 + b^2).$$

b) Origin of axes at corner:

$$I_x = \frac{bh^3}{3}, \quad I_y = \frac{hb^3}{3}, \quad I_{xy} = \frac{b^2 h^2}{4};$$

$$I_\rho = \frac{bh}{3}(h^2 + b^2),$$

$$I_{x_1} = \frac{b^3 h^3}{6(b^2 + h^2)}.$$

Regular polygon with *n* sides

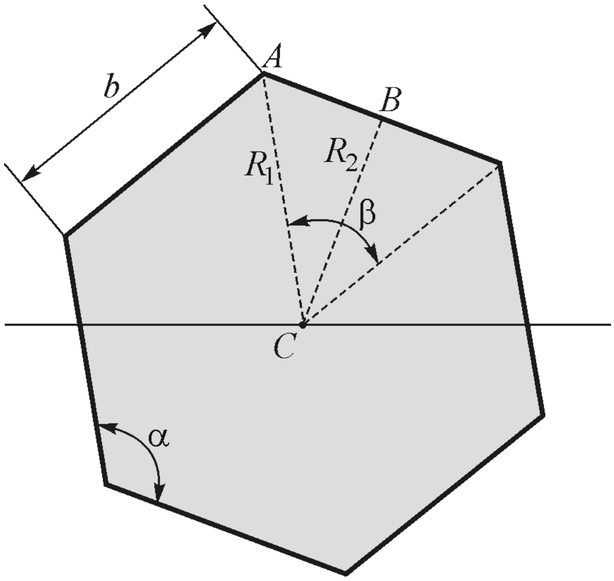


Fig. 73

Origin of axes at centroid:
C = centroid (at center of polygon),
n = number of sides ($n \geq 3$),
b = length of a side,
β = central angle for a side,
α = interior angle (or vertex angle),

$$\beta = \frac{360°}{n}, \quad \alpha = \left(\frac{n-2}{n}\right)180°,$$

$$\alpha + \beta = 180°;$$

R_1 = radius of circumscribed circle (line CA),
R_2 = radius of inscribed circle (line CB),

$$R_1 = \frac{b}{2}\csc\frac{\beta}{2}, \; R_2 = \frac{b}{2}\cot\frac{\beta}{2},$$

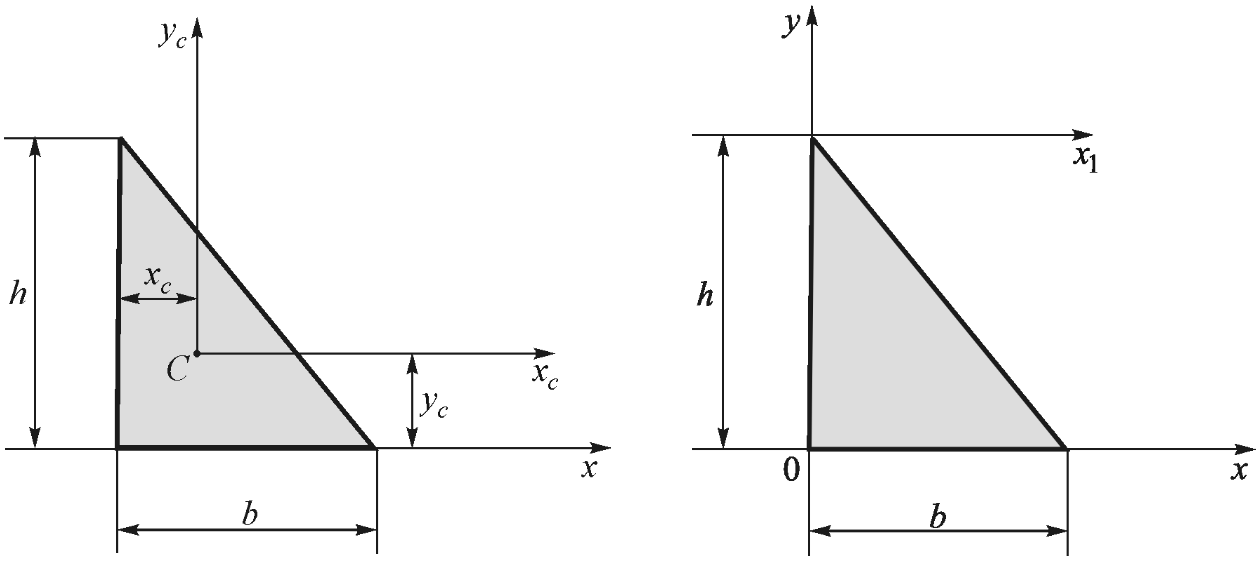
$$A = \frac{nb^2}{4}\cot\frac{\beta}{2};$$

I_c – moment of inertia about any axis through C (the centroid C is a principal point and every axis through C is a principal axis),

$$I_c = \frac{nb^3}{192}\left(\cot\frac{\beta}{2}\right)\left(3\cot^2\frac{\beta}{2}+1\right),$$

$$I_\rho = 2I_c.$$

Right triangle



a Fig. 74 b

a) Origin of axes at centroid:

$$A=\frac{bh}{2},\quad x_c=\frac{b}{3},\quad y_c=\frac{h}{3};$$

$$I_{x_c}=\frac{bh^3}{36},\quad I_{y_c}=\frac{hb^3}{36},$$

$$I_{x_c y_c}=-(b^2h^2)/72;$$

$$I_\rho=\frac{bh}{36}(h^2+b^2),\quad I_x=\frac{bh^3}{12}.$$

b) Origin of axes at vertex:

$$I_x=\frac{bh^3}{12},\quad I_y=\frac{hb^3}{12},\quad I_{xy}=\frac{b^2h^2}{24};$$

$$I_p=\frac{bh}{12}(h^2+b^2),\quad I_{x_1}=\frac{bh^3}{4}.$$

Semicircle

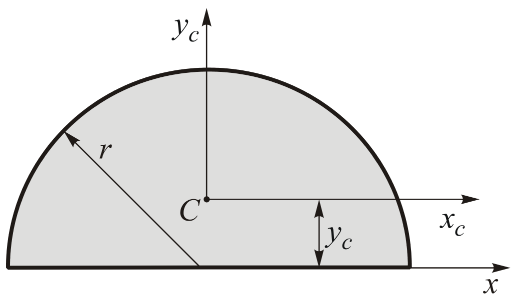


Fig. 75

Origin of axes at centroid:

$$A=\frac{\pi r^2}{2},\quad y_c=\frac{4r}{3\pi};$$

$$I_{x_c}=\frac{(9\pi^2-64)r^2}{72\pi}\approx 0.1098r^4,$$

$$I_{y_c}=\frac{\pi r^4}{8},$$

$$I_{xy_c}=I_{x_c y_c}=0,\quad I_x=\frac{\pi r^4}{8}.$$

Semisine wave

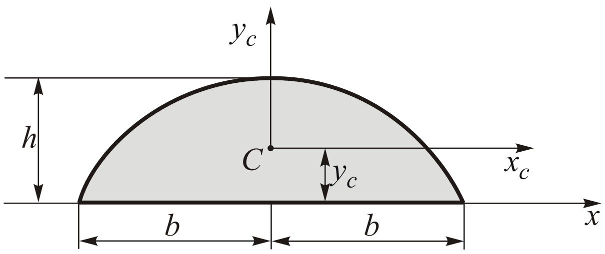


Fig. 76

Origin of axes at centroid:

$$A=\frac{4bh}{\pi},\quad y_c=\frac{\pi h}{8};$$

$$I_{x_c}=\left(\frac{8}{9\pi}-\frac{\pi}{16}\right)bh^3\approx 0.08659bh^3,$$

$$I_{y_c}=\left(\frac{4}{\pi}-\frac{32}{\pi^3}\right)hb^3\approx 0.2412hb^3,$$

$$I_{x_c y_c}=I_{xy_c}=0,\quad I_x=\frac{8bh^3}{9\pi}.$$

Thin circular arc

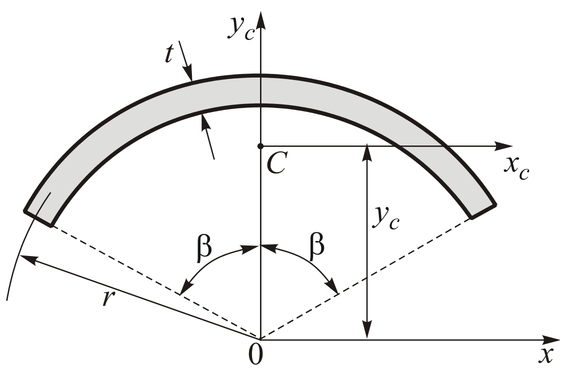


Fig. 77

Origin of axes at center of circle.
Approximate formulas for case when t is small:

β – angle in radians, $(\beta\le\pi/2)$;

$$A=2\beta rt,\quad y_c=(r\sin\beta)/\beta;$$

$$I_x=r^3t(\beta+\sin\beta\cos\beta),$$

$$I_{y_c}=r^3t(\beta-\sin\beta\cos\beta),$$

$$I_{x_c y_c}=I_{xy_c}=0,$$

$$I_{x_c}=r^3t\left(\frac{2\beta+\sin 2\beta}{2}-\frac{1-\cos 2\beta}{\beta}\right).$$

Note: For a semicircular arc, $(\beta=\pi/2)$.

Thin circular ring

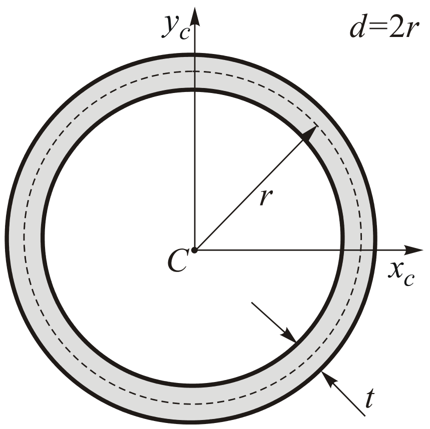


Fig. 78

Origin of axes at centroid.
Approximate formulas for case when t is small:

$$A = 2\pi rt = \pi dt\,,$$

$$I_{x_c} = I_{y_c} = \pi r^3 t = \frac{\pi d^3 t}{8}\,,$$

$$I_{x_c y_c} = 0\,,$$

$$I_\rho = 2\pi r^3 t = \frac{\pi d^3 t}{4}\,.$$

Thin rectangle

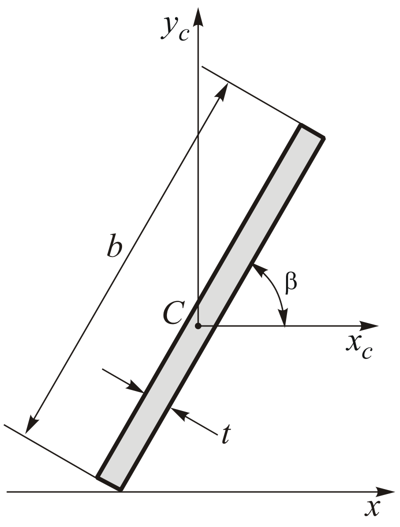


Fig. 79

Origin of axes at centroid.
Approximate formulas for case when t is small:

$$A = bt\,,$$

$$I_{x_c} = \frac{tb^3}{12}\sin^2\beta\,,$$

$$I_{y_c} = \frac{tb^3}{12}\cos^2\beta\,,$$

$$I_x = \frac{tb^3}{3}\sin^2\beta\,.$$

Trapezoid

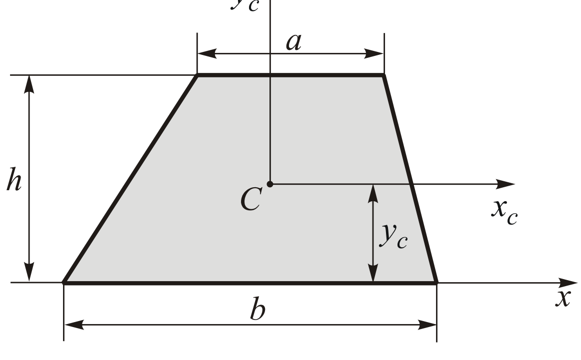


Fig. 80

Origin of axes at centroid:

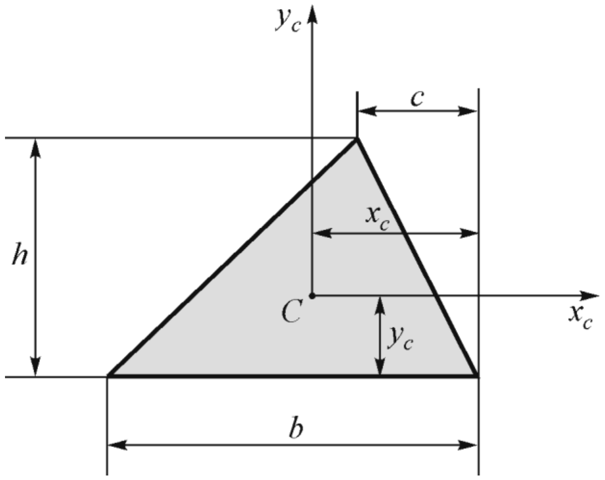
$$A = \frac{h(a+b)}{2}\,,$$

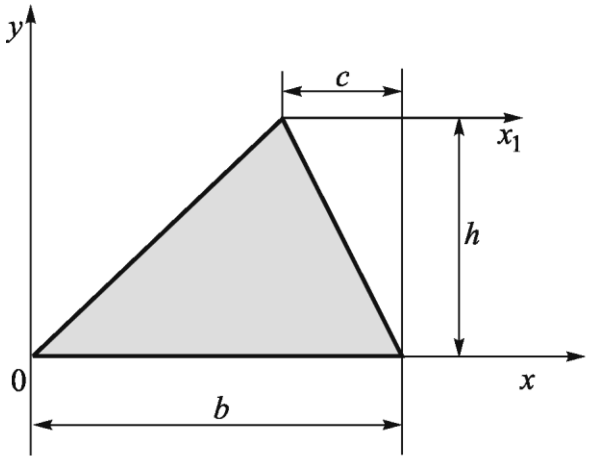
$$y_c = \frac{h(2a+b)}{3(a+b)}\,,$$

$$I_{x_c} = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}\,,$$

$$I_x = \frac{h^3(3a+b)}{12}\,.$$

Triangle





a Fig. 81 b

a) Origin of axes at centroid:

$$A = \frac{bh}{2}, \quad x_c = \frac{b+c}{3}, \quad y_c = \frac{h}{3};$$

$$I_{x_c} = \frac{bh^3}{36}, \quad I_{y_c} = \frac{bh}{36}(b^2 - bc + c^2);$$

$$I_{x_c y_c} = \frac{bh^2}{72}(b - 2c),$$

$$I_\rho = \frac{bh}{36}(h^2 + b^2 - bc + c^2).$$

b) Origin of axes at vertex:

$$I_x = \frac{bh^3}{3},$$

$$I_y = \frac{bh}{12}(3b^2 - 3bc + c^2),$$

$$I_{xy} = \frac{bh^2}{24}(3b - 2c),$$

$$I_{x_1} = \frac{bh^3}{4}.$$