LECTURE 2 Geometrical Properties of Rod Cross Sections (Part 2)

1 Moments of Inertia Transformation with Parallel Transfer of Axes.

Parallel-Axes Theorems

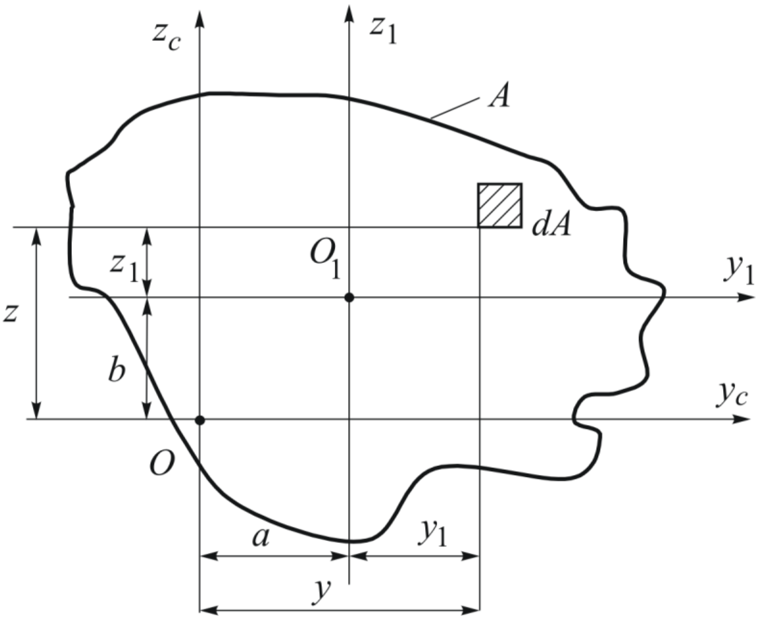


Fig. 1

Given: A, a, b, I_{y_c}, I_{z_c}, $I_{y_c z_c}$, where y_c and z_c are central axes, i.e. $S_{y_c} = S_{z_c} = 0$.

y_1 and z_1 are axes parallel to the y_c and z_c axes. The distance between z_1 and z_c axes is a and the distance between y_1 and y_c axes is b.

Determine: the moments of inertia with respect to z_1 and y_1 axes.

By definition

$$I_{y_1} = \int_A z_1^2 dA, \quad I_{z_1} = \int_A y_1^2 dA, \quad I_{y_1 z_1} = \int_A y_1 z_1 dA. \tag{1}$$

In Fig.1 it is seen, that

$$z_1 = z - b, \quad y_1 = y - a. \tag{2}$$

Substituting z_1 and y_1 from expressions (2) into formula (1), we find

$$I_{y_1} = \int_A (z-b)^2 dA = \int_A z^2 dA - 2b \int_A z dA + b^2 \int_A dA, \tag{3}$$

$$I_{z_1} = \int\limits_A (y-a)^2 dA = \int\limits_A y^2 dA - 2a\int\limits_A y dA + a^2 \int\limits_A dA, \quad (4)$$

$$I_{y_1 z_1} = \int\limits_A (y-a)(z-b) dA = \int\limits_A yz dA - a\int\limits_A z dA - b\int\limits_A y dA + ab\int\limits_A dA. \quad (5)$$

If z_c and y_c axes are central, then $S_{y_c} = S_{z_c} = 0$ and obtained expressions are significantly simplified

$$\begin{aligned} I_{y_1} &= I_{y_c} + b^2 A, \\ I_{z_1} &= I_{z_c} + a^2 A, \\ I_{y_1 z_1} &= I_{y_c z_c} + abA. \end{aligned} \quad \text{– parallel-axes theorem.} \quad (6)$$

The moment of inertia of section with respect to an arbitrary axis in its plane is equal to the moment of inertia with respect to parallel centroidal axis plus the product of the area and the square of the distance between two axes.

The product of inertia of a section with respect to an arbitrary pair of axes in its plane is equal to the product of inertia with respect to parallel centroidal axes plus the product of a section area and the coordinates of the centroid with respect to the pair of axes.

It follows from the first two formulas of (6) that in the family of parallel axes the moment of inertia with respect to the central axis is a minimum.

While determining the product of inertia by formulas (6) it is necessary to take into account the signs of values *a* and *b*. They are centroid coordinates *O* in z_1Oy_1 orthogonal system.

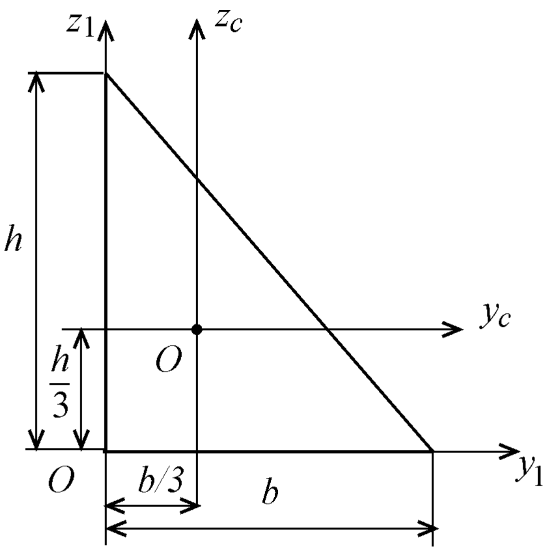


Fig. 2

***Example 1** Determine axial moments of inertia and product of inertia for the right triangle relative to central axes which are parallel to triangle legs.*

Given: h, b, $I_{y_1} = bh^3/12$, $I_{z_1} = hb^3/12$, $I_{y_1 z_1} = +b^2h^2/24$ (they are found by integration)

Determine: I_{y_c}, I_{z_c}, $I_{y_c z_c}$.

Let us use the results of previous example and parallel axes transfer formulae.

In this case, the transfer from an arbitrary y_1, z_1 axes to central $y_c z_c$ axes is necessary to perform:

$$I_{y_c} = I_{y_1} - \left(\frac{h}{3}\right)^2 A,$$
$$I_{z_c} = I_{z_1} - \left(\frac{b}{3}\right)^2 A,$$
$$I_{y_c z_c} = I_{y_1 z_1} - \left(+\frac{h}{3}\right)\left(+\frac{b}{3}\right) A.$$

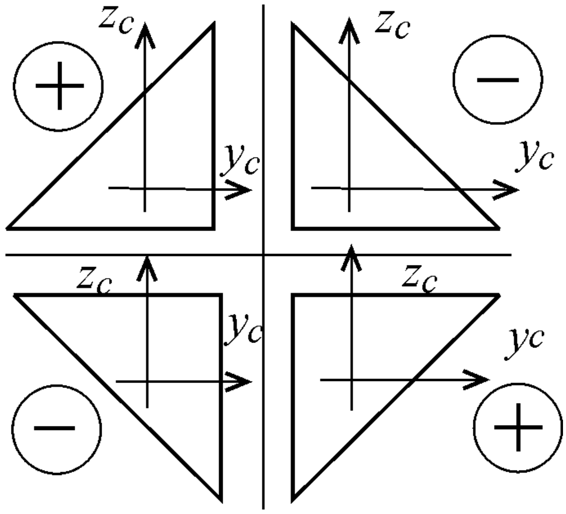


Fig. 3

After substitutions and simplifications we get in result:

$$I_{y_c}^{\triangle} = \frac{bh^3}{36}, \qquad I_{z_c}^{\triangle} = \frac{hb^3}{36}, \qquad I_{y_c z_c}^{\triangle} = -\frac{b^2 h^2}{72}.$$

Note: sign of the product $I_{y_c z_c}^{\triangle}$ depends on orientation of the triangle relative to selected orthogonal system of coordinates.

Example 2 Calculate the product of inertia $I_{x_c y_c}$ of the *Z*-section shown in Fig 4.

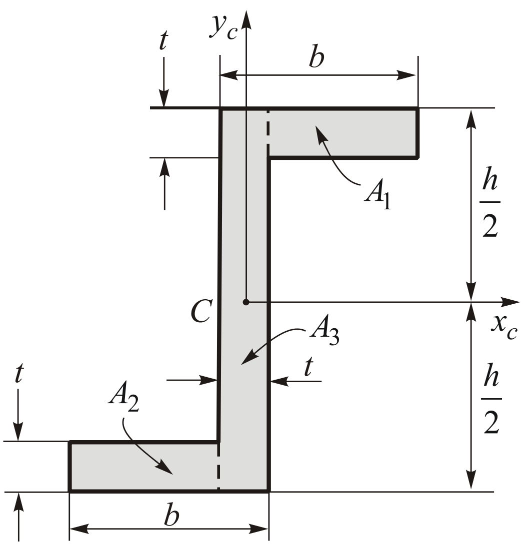


Fig. 4

The section has width *b*, height *h*, and thickness *t*. To obtain the product of inertia with respect to the *x*, *y* axes through the centroid, we divide the area into three parts and use the parallel-axis theorem. The parts are as follows: (1) a rectangle of width $b-t$ and thickness *t* in the upper flange, (2) a similar rectangle in the lower flange, and (3) a web rectangle with height *h* and thickness *t*.

The product of inertia of the web rectangle with respect to the *x*, *y* axes is zero (from symmetry).

The product of inertia $\left(I_{xy}\right)_1$ of the upper flange rectangle (with respect to the x_c, y_c axes) is determined by using the parallel-axis theorem:

$$(I_{xy})_1 = I_{x_c y_c} + A d_1 d_2,$$

in which $I_{x_c y_c}$ is the product of inertia of the rectangle with respect to its own centroid, A is the area of the rectangle, d_1 is the x coordinate of the centroid of the rectangle, and d_2 is the y coordinate of the centroid of the rectangle. Thus,

$$I_{x_c y_c} = 0, \quad A = (b-t)t, \quad d_1 = \frac{h}{2} - \frac{t}{2}, \quad d_2 = \frac{b}{2};$$

and the product of inertia of the upper flange rectangle is

$$(I_{xy})_1 = I_{x_c y_c} + A d_1 d_2 = 0 + t(b-t)\left(\frac{h}{2} - \frac{t}{2}\right)\left(\frac{b}{2}\right) = \frac{bt}{4}(h-t)(b-t).$$

The product of inertia of the lower flange rectangle is the same. Therefore, the product of inertia of the entire *Z*-section is twice $\left(I_{xy}\right)_1$, or

$$I_{xy} = \frac{bt}{2}(h-t)(b-t).$$

Note: This product of inertia is positive because the flanges lie in the first and third quadrants.

Example 3 Determine centroidal axial moments of inertia of a parabolic semisegment

The parabolic semisegment *OAB* shown in Fig. 5 has base *b* and height *h*. Using the parallel-axis theorem, determine the moments of inertia I_{x_c} and I_{y_c} with respect to the centroidal axes x_c and y_c.

We can use the parallel-axis theorem (rather than integration) to find the centroidal moments of inertia because we already know the area *A*, the centroidal coordinates x_c and y_c, and the moments of inertia I_x and I_y with respect to the *x* and *y* axes. These quantities may be obtained by integration (see axial moment of inertia of a parabolic semisegment). They are repeated here:

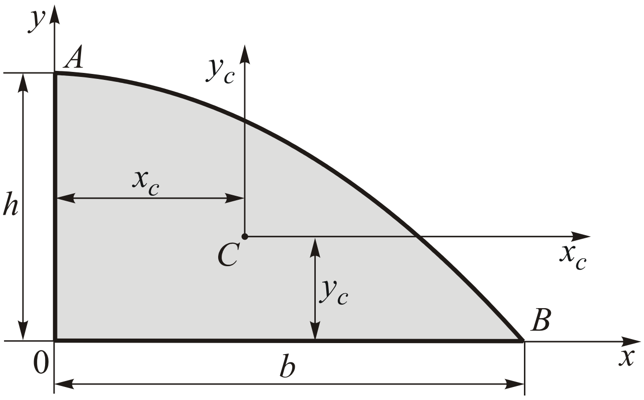


Fig. 5

$$A = \frac{2bh}{3}, \quad x_c = \frac{3b}{8}, \quad y_c = \frac{2h}{5},$$

$$I_x = \frac{16bh^3}{105}, \quad I_y = \frac{2hb^3}{15}.$$

To obtain the moment of inertia with respect to the x_c axis, we write the parallel-axis theorem as follows:

$$I_x = I_{x_c} + Ay_c^2.$$

Therefore, the moment of inertia I_{x_c} is

$$I_{x_c} = I_x - Ay_c^2 = \frac{16bh^3}{105} - \frac{2bh}{3}\left(\frac{2h}{5}\right)^2 = \frac{8bh^3}{175}.$$

In a similar manner, we obtain the moment of inertia with respect to the y_c axis:

$$I_{y_c} = I_y - Ax_c^2 = \frac{2hb^3}{15} - \frac{2bh}{3}\left(\frac{3b}{8}\right)^2 = \frac{19hb^3}{480}.$$

Example 4 Determine the moment of inertia I_{x_c} with respect to the horizontal axis x_c through the centroid C of the beam cross section shown in Fig. 6.

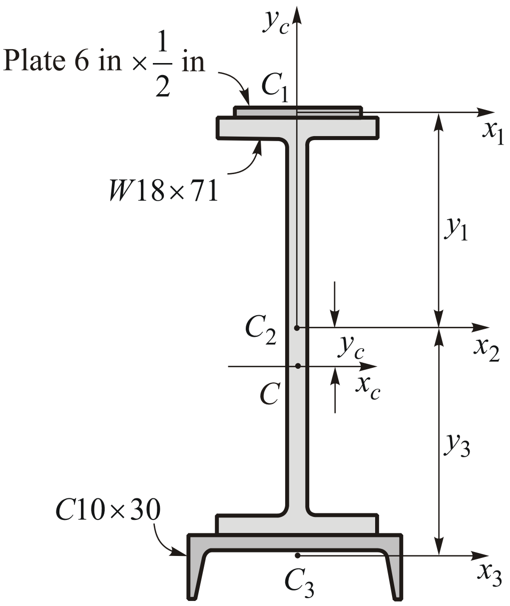


Fig. 6

The position of the centroid C was determined previously and equals to $y_c = 1.8$ in.

Note: It will be clear from beam theory that axis x_c is the neutral axis for bending of this beam, and therefore the moment of inertia I_{x_c} must be determined in order to calculate the stresses and deflections of this beam.

We will determine the moment of inertia I_{x_c} with respect to axis x_c by applying the **parallel-axis theorem** to each individual part of the composite area. The area is divided naturally into three parts: (1) the cover plate, (2) the wide-flange section, and (3) the channel section. The following areas and centroidal distances were obtained previously:

$$A_1 = 3.0\,\text{in.}^2, \quad A_2 = 20.8\,\text{in.}^2, \quad A_3 = 8.82\,\text{in.}^2;$$

$$y_1 = 9.485\,\text{in.}, \quad y_2 = 0, \quad y_3 = 9.884\,\text{in.}, \quad y_c = 1.80\,\text{in.}$$

The moments of inertia of the three parts with respect to horizontal axes through their own centroids C_1, C_2, and C_3 are as follows:

$$I_1 = \frac{bh^3}{12} = \frac{1}{12}(6.0\,\text{in.})(0.5\,\text{in.})^3 = 0.063\,\text{in.}^4;$$

$$I_2 = 1170\,\text{in.}^4; \quad I_3 = 3.94\,\text{in.}^4$$

Now we can use the parallel-axis theorem to calculate the moments of inertia about axis x_c for each of the three parts of the composite area:

$$I_{x_c}^{I} = I_1 + A_1(y_1 + y_c)^2 = 0.063\,\text{in.}^4 + (3.0\,\text{in.}^2)(11.28\,\text{in.})^2 = 382\,\text{in.}^4;$$

$$I_{x_c}^{II} = I_2 + A_2 y_c{}^2 = 1170\,\text{in.}^4 + (20.8\,\text{in.}^2)(1.80\,\text{in.})^2 = 1240\,\text{in.}^4;$$

$$I_{x_c}^{III} = I_3 + A_3(y_3 - y_c)^2 = 3.94\,\text{in.}^4 + (8.82\,\text{in.}^2)(8.084\,\text{in.})^2 = 580\,\text{in.}^4.$$

The sum of these individual moments of inertia gives the moment of inertia of the entire cross-sectional area about its centroidal axis x_c:

$$I_{x_c} = I_{x_c}^{I} + I_{x_c}^{II} + I_{x_c}^{III} = 2200\,\text{in.}^4.$$

2 Moments of Inertia Change and Coordinate Axes Rotating

Let us consider a cross section of a rod. Relate it to a system of coordinates $z_1 O y_1$.

Isolate an element dA from the area A with coordinates z, y. Let us consider that cross section’s axial moments of inertia I_y, I_z and product of inertia I_{yz} are given (Fig. 7):

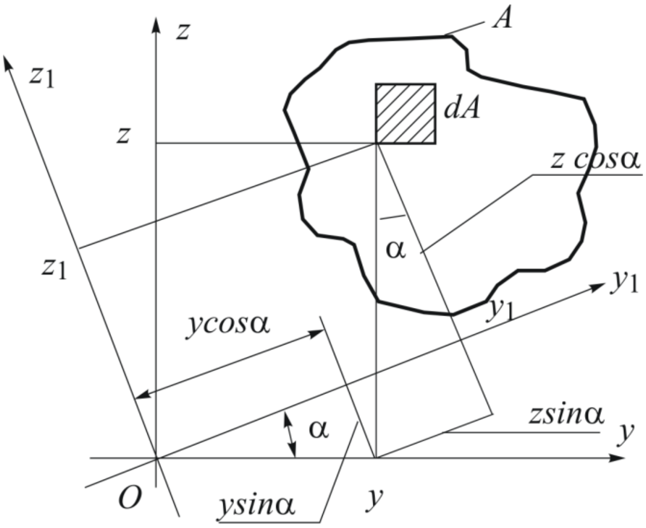


Fig. 7

It is required to determine I_y, I_z, I_{yz}, i.e. the moments of inertia with respect to axes y_1, z_1 rotated through an angle α in relation to system y, z ($\alpha > 0$ i.e. counter clockwise rotation is chosen as positive).

It should be observed here that the ***point O is not the section centroid***.

Using Fig.7, we find:

$$z_1 = z\cos\alpha - y\sin\alpha, \quad y_1 = y\cos\alpha + z\sin\alpha. \tag{7}$$

By definition

$$I_{y_1} = \int_A z_1^2 dA, \; I_{z_1} = \int_A y_1^2 dA, \; I_{y_1 z_1} = \int_A z_1 y_1 dA. \tag{8}$$

Then

$$I_{y_1} = \int_A (z\cos\alpha - y\sin\alpha)^2 dA = \cos^2\alpha \int_A z^2 dA -$$

$$-2\sin\alpha\cos\alpha \int_A yzdA + \sin^2\alpha \int_A y^2 dA. \tag{9}$$

By similar way

$$I_{z_1} = \int_A (y\cos\alpha + z\sin\alpha)^2 dA = \cos^2\alpha \int_A y^2 dA +$$

$$+2\sin\alpha\cos\alpha \int_A yzdA + \sin^2\alpha \int_A z^2 dA, \tag{10}$$

$$I_{y_1 z_1} = \int_A (z\cos\alpha - y\sin\alpha)(y\cos\alpha + z\sin\alpha) dA =$$

$$= \cos^2\alpha \int_A yzdA - \sin^2\alpha \int_A yzdA - \sin\alpha\cos\alpha \int_A y^2 dA + \sin\alpha\cos\alpha \int_A z^2 dA =$$

$$= \left(\cos^2\alpha - \sin^2\alpha\right) \int_A yxdA + 2\sin\alpha\cos\alpha \frac{\int_A z^2 dA - \int_A y^2 dA}{2}. \tag{11}$$

Using moment of inertia definition and the formula $\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$, we may write, that

$$\begin{aligned} I_{y_1} &= I_y \cos^2\alpha - I_{yz}\sin 2\alpha + I_z \sin^2\alpha, \\ I_{z_1} &= I_z \cos^2\alpha + I_{yz}\sin 2\alpha + I_y \sin^2\alpha, \\ I_{y_1 z_1} &= I_{yz}\cos 2\alpha + \frac{I_y - I_z}{2}\sin 2\alpha. \end{aligned} \tag{12}$$

We note that functions (9, 10, 11) are periodic with a period π. The axial moments of inertia are positive. ***They can be minimum or maximum but simultaneously***, at the same angle α_p. The product of inertia changes its sign in rotation axes.

3 Sum of Axial Moments of Inertia

It is evident, that

$$\begin{aligned} I_{y_1} + I_{z_1} &= I_y \cos^2\alpha + I_z \sin^2\alpha - I_{yz}\sin 2\alpha + I_y \sin^2\alpha + \\ &+ I_z \cos^2\alpha + I_{yz}\sin 2\alpha = I_y\left(\cos^2\alpha + \sin^2\alpha\right) + I_z\left(\sin^2\alpha + \cos^2\alpha\right). \end{aligned} \tag{13}$$

Thus the ***sum of the axial moments of inertia with respect to two mutually perpendicular axes depends on the angle of rotation and remains constant when the axes are rotated***.

Note, that

$$y^2 + z^2 = \rho^2 \tag{14}$$

and

$$\int_A \left(y^2 + z^2\right) dA = \int_A \rho^2 dA, \tag{15}$$

where ρ is the distance from the origin to the element of area.

Thus

$$I_y + I_z = I_\rho, \tag{16}$$

where I_ρ is the familiar polar moment of inertia.

4 Principal Axes. Principal Central Axes. Principal Moments of Inertia

Each of the quantities I_{y_1} and I_{z_1} changes with the axis α rotation angle, but their sum remains unchanged. Consequently there exists an angle α_p at which one of moments of inertia attains its maximum value while the other assumes a minimum value.

Differentiating the expression for I_{y_1} (9) with respect to α and equating the derivative to zero, we find

$$\frac{dI_{y1}}{d\alpha} = 2I\cos\alpha_p\sin\alpha_p - 2I_{yz}\cos 2\alpha_p + 2I_Z\cos\alpha_p\sin\alpha_p = 0,$$

$$\left(I_z - I_y\right)\sin 2\alpha_p = 2I_{yz}\cos 2\alpha_p, \tag{17}$$

$$\tan 2\alpha_p = \frac{2I_{yz}}{I_z - I_y}.$$

For this value of the angle α_p, one of axial moments is maximum and the other one is minimum.

If $\tan 2\alpha_p > 0$, then axes should be rotated counter clockwise.

For the same angle α_p, the product of inertia vanishes:

$$I_{y_1 z_1} = \frac{I_z - I_y}{2}\sin 2\alpha_p + \frac{I_y - I_z}{2}\sin 2\alpha_p \equiv 0. \tag{18}$$

***Axes, with respect to which the product of inertia is zero and the axial moments takes extremal values, are called* principal axes**.

If, in addition, they are central, such axes called **principal central axes**.

***The axial moments of inertia with respect to principal axes are called* principal moments of inertia**.

Principal moments of inertia are determined by using the following formulas:

$$I_{\substack{\max\\ \min}} = \frac{I_z + I_y}{2} \pm \left(\frac{I_z - I_y}{2} \frac{I_z - I_y}{\sqrt{\left(I_z - I_y\right)^2 + 4I_{yz}^2}} + \frac{2I_{yz}^2}{\sqrt{\left(I_z - I_y\right)^2 + 4I_{yz}^2}} \right),$$

$$I_{\substack{\max \\ \min}} = \frac{I_z + I_y}{2} \pm \frac{1}{2}\sqrt{\left(I_z - I_y\right)^2 + 4I_{yz}^2} \tag{19}$$

The upper sign corresponds to the maximal moment of inertia and the lower one – to the minimal moment.

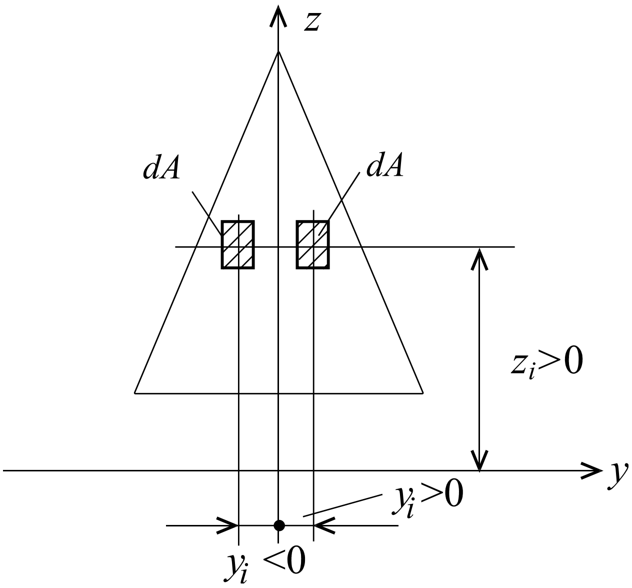


Fig. 8

If section has an axis of symmetry, this axis is by all means the principal one. It means that sectional parts, lying on different sides of the axis, and products of inertia are equal, but have opposite signs. Consequently, $I_{yz} = 0$ and y and z axes are principal

$$I_{yz}^{\triangle} = \sum_{i=1}^{2} I_{yz}^{\triangleright} = 0 .$$

Example 5 Determine the orientations of the principal centroidal axes and the magnitudes of the principal centroidal moments of inertia for the cross-sectional area of the *Z*-section shown in Fig. 9.

Given: Use the following numerical data: height h = 200 mm, width b = 90 mm, and thickness t = 15 mm.

Let us use the x_c, y_c axes as the reference axes through the centroid C. The moments and product of inertia with respect to these axes can be obtained by dividing the area into three rectangles and using the parallel-axis theorems. The results of such calculations are as follows:

$$I_{x_c} = 29.29\times10^6 \text{ mm}^4,\ \ I_{yc} = 6.667\times10^6 \text{ mm}^4,\ \ I_{x_c y_c} = -9.366\times10^6 \text{ mm}^4.$$

Substituting these values into the equation for the angle θ_p Eq. (17), we get

$$\tan 2\theta_p = -\frac{2I_{xy}}{I_x - I_y} = 0.7930,\ \ 2\theta_p = 38.4° \text{ and } 218.4° .$$

Thus, the two values of θ_p are

$$\theta_p = 19.2° \text{ and } 109.2°.$$

Using these values of θ_p in the transformation equation for I_{x_1}

$$I_{x_1} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta_p - I_{xy}\sin 2\theta_p \tag{20}$$

we find $I_{x_1} = 32.6\times10^6$ mm^4 and $I_{x_1} = 2.4\times10^6$ mm^4, respectively. The same values are obtained if we substitute into equations:

$$I_U = I_{max} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$
$$I_V = I_{min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{21}$$

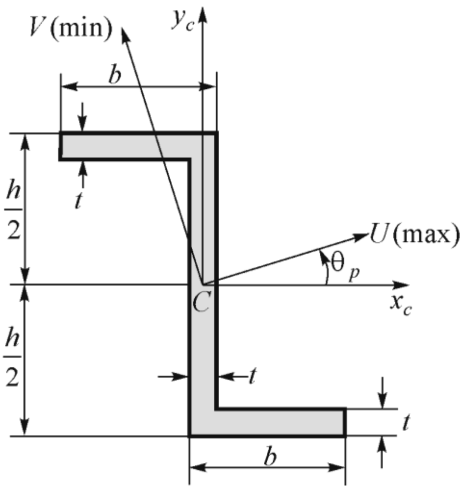


Fig. 9

Thus, the principal moments of inertia and the angles to the corresponding principal axes are:

$$I_U = 32.6\times10^6 \text{ mm}^4, \quad \theta_{p_1} = 19.2°;$$

$$I_V = 2.4\times10^6 \text{ mm}^4, \quad \theta_{p_2} = 109.2°.$$

The principal axes are shown in Fig. 9 as the *U*, *V* axes.

Example 6 Determine the orientations of the principal centroidal axes and the magnitudes of the principal centroidal moments of inertia for the cross-sectional area shown in Fig. 10. Use the following numerical data (see table).

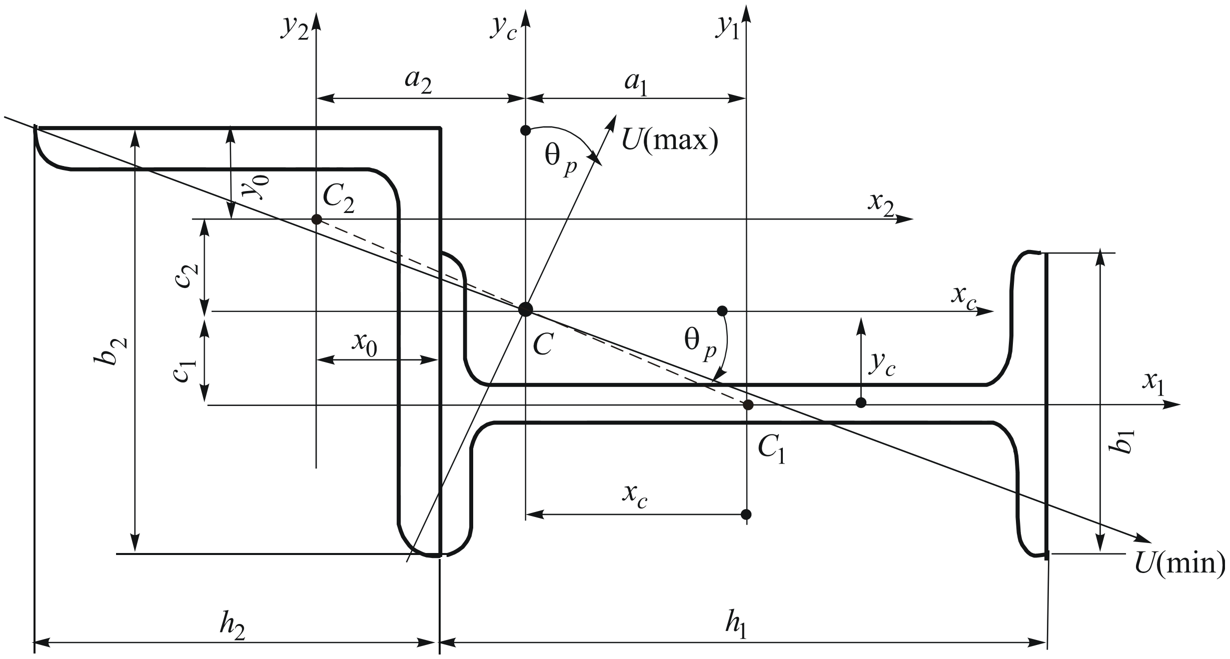


Fig. 10

Parts of the composite area	Geometrical properties						
	h_i, m	b_i, m	A_i, m^2	I_{x_i}, m^2	I_{y_i}, m^4	$I_{x_i y_i}$, m^4	y_0, m
1- $\vdash\!\dashv$	0.2	0.1	26.8×10^{-4}	115×10^{-8}	1840×10^{-8}	0	-
2- $\neg$	0.16	0.16	31.4×10^{-4}	774×10^{-8}	774×10^{-8}	-445×10^{-8}	4.3×10^{-2}

The coordinates of angular section centroid C_2 are known from assortment ($x_0 = y_0 = 4.3\times10^{-2}$ m).

The coordinates of the centroid C are determined beforehand and equals to:

$$x_c = -7.715\times10^{-2}\text{ m},$$

$$y_c = 3.615\times10^{-2}\text{ m}.$$

Note: the first element (I-beam) was chosen as original in this calculation.

Let us use the x_c, y_c axes as the reference axes through the centroid C. The moments and product of inertia with respect to these axes can be obtained using the **parallel-axis theorems**. The results of such calculations are as follows.

$$I_{x_c}^{\vdash\!\dashv} = I_{x_c} + I_{x_c}^{\neg}, \tag{22}$$

$$I_{x_c}^{\vdash\!\dashv} = I_{x_1}^{\vdash\!\dashv} + c_1^2 A_1 = 115\times10^{-8} + 3.615^2\times26.8\cdot10^{-8} = 465.23\times10^{-8}\text{ m}^4,$$

$$I_{x_c}^{\neg} = I_{x_2}^{\neg} + c_2^2 A_2 = 774\times10^{-8} + 3.085^2\times31.4\times10^{-8} = 1072.8\times10^{-8}\text{ m}^4,$$

$$I_{x_c} = \left(465.23 + 1072.8\right)\times10^{-8} = 1538\times10^{-8}\text{ m}^4,$$

$$I_{y_c}^{\vdash\!\dashv} = I_{y_c}^{\neg} + I_{y_c}, \tag{23}$$

$$I_{y_c}^{\vdash\!\dashv} = I_{y_1}^{\vdash\!\dashv} + a_1^2 A_1 = 1840\times10^{-8} + 7.715^2\times26.8\times10^{-8} = 3435.2\times10^{-8}\text{ m}^4,$$

$$I_{y_c}^{\neg} = I_{y_2}^{\neg} + a_2^2 A_2 = 774\times10^{-8} + 6.585^2\times31.4\times10^{-8} = 2135.6\times10^{-8}\text{ m}^4,$$

$$I_{y_c} = \left(3435.2 + 2135.6\right)\times10^{-8} = 5570.8\times10^{-8}\text{ m}^4,$$

$$I_{x_c y_c}^{\vdash\dashv} = I_{x_c y_c} + I_{x_c y_c}^{\neg}, \qquad (24)$$

$$I_{x_c y_c} = I_{x_1 y_1} + a_1 c_1 A_1 = 0 + 7.715(-3.615)\times 10^{-4} \times 26.8 \times 10^{-4} = -747.4\times 10^{-8}\ \text{m}^4,$$

$$I_{x_c y_c}^{\neg} = I_{x_2 y_2}^{\neg} + a_2 c_2 A_2.$$

In last equation the product of inertia $I_{y_2 z_2}^{\neg}$ is unknown. Let's determine it using Fig. 11. Note, that y_3, z_3 axes are really principal axes of the section enertia since y_3 is the axis of its simmetry. The values of principal central moments of inertia of the angle are specified in assortment. Let us select these values:

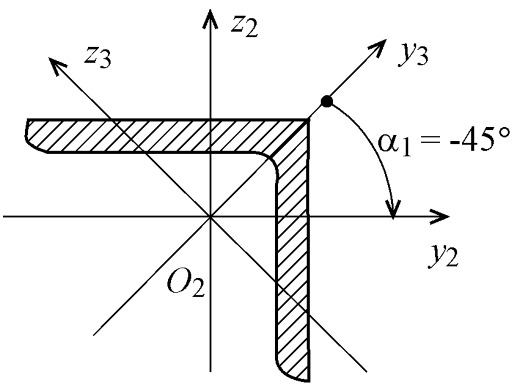


Fig. 11

$$I_{y_3}^{\neg} = I_{\max}^{\neg} = 1229 \times 10^{-8}\ \text{m}^4;$$

$$I_{z_3}^{\neg} = I_{\min}^{\neg} = 319 \times 10^{-8}\ \text{m}^4.$$

Note: If in assortment only one of these two values is specified, the second one may be calculated using the fact that the sum of two axial moments of inertia is unchanged in axes rotation:

$$I_{\max}^{\neg} + I_{\min}^{\neg} = I_{y_2}^{\neg} + I_{z_2}^{\neg}. \qquad (25)$$

It is also evident, that $I_{y_3 z_3}^{\neg} \equiv 0$.

Applying rotational formula for product of inertia we find

$$I_{y_2 z_2}^{\neg} = I_{y_3 z_3}^{\neg}\cos 2\alpha_1 + \frac{I_{y_3}^{\neg} - I_{z_3}^{\neg}}{2}\sin 2\alpha_1 = 0 + \frac{I_{\max}^{\neg} - I_{\min}^{\neg}}{2}\sin(-90^\circ) =$$

$$= \frac{1229 - 319}{2} 10^{-8}(-1) = -455 \times 10^{-8}\ \ \text{m}^4.$$

Consequently

$$I_{x_c y_c}^{\neg} = -455 \times 10^{-8} + (-6.585)3.085 \times 31.4 \times 10^{-8} = -1092.9 \times 10^{-8}\,\text{m}^4.$$

After substitutions the result is

$$I_{x_c y_c} = (-747.4 - 1092.9)10^{-8} = -1840.3 \times 10^{-8} \text{ m}^4.$$

Substituting these values into the equation for the angle θ_p, we get

$$tg 2\theta_p = \frac{2I_{x_c y_c}}{I_{y_c} - I_{x_c}} = \frac{-2 \times 1840.3}{5570.8 - 1538} = -0.9127 \Rightarrow 2\theta_p = -42^\circ 24' \Rightarrow \theta_p = -21^\circ 12'.$$

The principal moments of inertia are

$$I_{\substack{U \\ V}} = I_{\substack{\max \\ \min}} = \frac{I_{x_c} + I_{y_c}}{2} \pm \sqrt{\left(\frac{I_{x_c} - I_{y_c}}{2}\right)^2 + I_{x_c y_c}^2} = (3554.4 \pm 2293.2)10^{-8} \text{ m}^4,$$

$$I_U = I_{\max} = 5847.6 \times 10^{-8} \text{ m}^4, \quad I_V = I_{\min} = 1261.2 \times 10^{-8} \text{ m}^4.$$

Checking the results:

a) $I_{\max} > I_{y_c} > I_{x_c} > I_{\min}$,

$$5847.6 \times 10^{-8} > 5570.8 \times 10^{-8} > 1538 \times 10^{-8} > 1261.2 \times 10^{-8};$$

b) $I_{\max} + I_{\min} = I_{x_c} + I_{y_c}$,

$$5847.6 \times 10^{-8} + 1261.2 \times 10^{-8} = 5570.8 \times 10^{-8} + 1538 \times 10^{-8},$$

$$\left(7108.8 \times 10^{-8} = 7108.8 \times 10^{-8}\right);$$

c) $$I_{UV} = I_{x_c y_c} \cos 2\theta_p + \frac{I_{y_c} - I_{z_c}}{2} \sin 2\theta_p =$$

$$= \left[-1804.3 \times 0.7384 + \frac{1538 - 5570.8}{2} \times (-0.6743)\right] \times 10^{-8} = (-1358.9 + 1359) \times 10^{-8} \text{ m}^4 \cong 0.$$

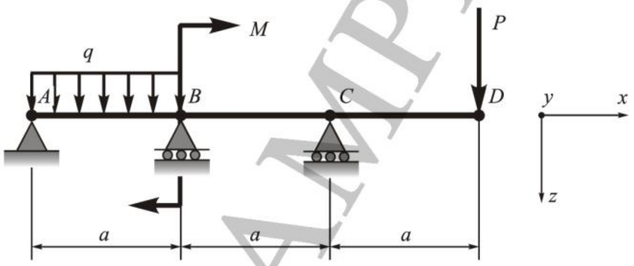
5 Example of home problem

National aerospace university
“Kharkiv Aviation Institute”
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: Internal Forces in Multispan Beams.

Full name of the student, group

Variant: 1 **Complexity: 1**



Given: $q = 10$ kN/m; $P = 20$ kN; $M = 10$ kNm; $a = 3$ m.

Goal:
1) open static indeterminacy using the equation of three moments and draw the graphs $Q_z(x)$, $M_y(x)$.

signature
Full name of the lecturer

Mark: [ ]

Solution

1. Use the following numerical data (see Table) and draw the section in scale (Fig. 1).

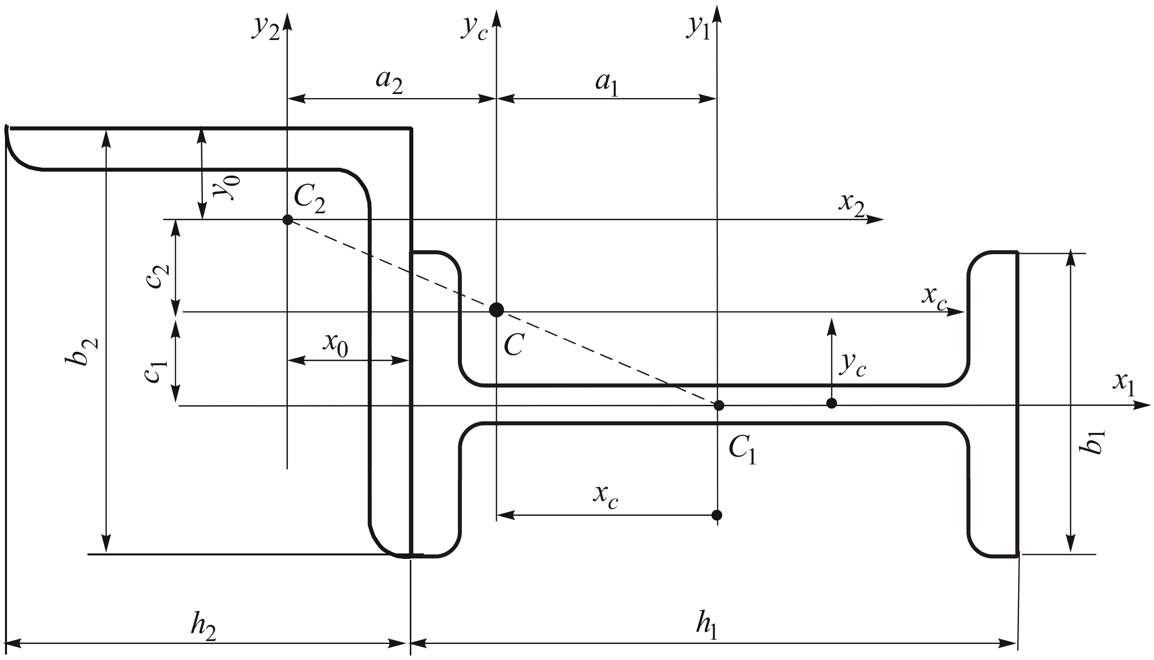


Fig. 1

Table

Parts of the composite area	Geometrical properties								
	h_i, m	b_i, m	A_i, m^2	I_{x_i}, m^4	I_{y_i}, m^4	$I_{x_i y_i}$, m^4	$I_{\max_i}$, m^4	$I_{\min_i}$, m^4	y_0, m
1 – GOST 8239-72	**0.2**	**0.1**	**26.8×10^{-4}**	**115×10^{-8}**	**1840×10^{-8}**	**0**	**1840×10^{-8}**	**115×10^{-8}**	**–**
2 – GOST 8509-72	**0.16**	**0.16**	**31.4×10^{-4}**	**774×10^{-8}**	**774×10^{-8}**	**–**	**1229×10^{-8}**	**319×10^{-8}**	**4.3×10^{-2}**

The coordinates of two C_1 and C_2 centroids for the parts are known from assortments ($x_0 = y_0 = 4.3\times10^{-2}$ m).

2. Calculation of the centroidal coordinates for composite area.

Axes x_1, y_1 are selected as reference axes in this study (see Fig. 1).

The following formulae are used

$$x_c = S_{y_1}/A, \quad y_c = S_{x_1}/A, \quad \text{where}$$

$$S_{y_1} = S_{y_1}^{\mathrm{I}} + S_{y_1}^{\urcorner}; \quad S_{x_1} = S_{x_1}^{\mathrm{I}} + S_{x_1}^{\urcorner}.$$

$$A = A^{\mathrm{I}} + A^{\urcorner} = 26.8\times10^{-4} + 31.4\times10^{-4} = 58.2\times10^{-4}\ \text{m}^2.$$

$S_{x_1}^{\mathrm{I}}$ and $S_{y_1}^{\mathrm{I}}$ are zero due to central character of x_1, y_1 axes for I-beam.

$$S_{x_1}^{\urcorner} = A^{\urcorner}(+(b_2 - \frac{b_1}{2} - y_0)) = 31.4\times10^{-4}(+(0.16-0.05-0.043)) = 31.4\times10^{-4}\times0.067 =$$
$$= +2.10\times10^{-4}\ \text{m}^3.$$

$$S_{y_1}^{\urcorner} = A^{\urcorner}(-(\frac{h_1}{2} + x_0)) = 31.4\times10^{-4}(-(0.1+0.043)) = -4.49\times10^{-4}\ \text{m}^3.$$

$$S_{y_1} = 0 - 4.49\times10^{-4} = -4.49\times10^{-4}\ \text{m}^3.$$

$$S_{x_1} = 0 + 2.10\times10^{-4} = +2.10\times10^{-4}\ \text{m}^3.$$

$$x_c = -4.49\times10^{-4}/58.2\times10^{-4} = -0.077\ \text{m} = -7.715\ \text{cm}.$$

$$y_c = +2.10\times10^{-4}/58.2\times10^{-4} = +0.03615\ \text{m} = +3.615\ \text{cm}.$$

Results: the coordinates of the C centroid are equal to:

$$x_c = -7.715\times10^{-2}\ \text{m},$$
$$y_c = 3.615\times10^{-2}\ \text{m}.$$

They are shown on Fig. 1.

3. Calculation of central moments of inertia relative to central x_c, y_c axes.

Let us denote the x_c, y_c axes as the centroidal axes of the composite area. The moments and product of inertia with respect to these axes can be obtained using the parallel-axis theorems. The results of such calculations are as follows.

$$I_{x_c} = I_{x_c}^{\vdash\dashv} + I_{x_c}^{\neg},$$

$$I_{x_c}^{\vdash\dashv} = I_{x_1}^{\vdash\dashv} + c_1^2 A_1 = 115\times10^{-8} + 3.615^2\times26.8\times10^{-8} = 465.23\times10^{-8}\,\text{m}^4,$$

$$I_{x_c}^{\neg} = I_{x_2}^{\neg} + c_2^2 A_2 = 774\times10^{-8} + 3.085^2\times31.4\times10^{-8} = 1072.8\times10^{-8}\,\text{m}^4,$$

$$I_{x_c} = (465.23 + 1072.8)10^{-8} = 1538\times10^{-8}\,\text{m}^4.$$

$$I_{y_c} = I_{y_c}^{\vdash\dashv} + I_{y_c}^{\neg},$$

$$I_{y_c}^{\vdash\dashv} = I_{y_1}^{\vdash\dashv} + a_1^2 A_1 = 1840\times10^{-8} + 7.715^2\times26.8\times10^{-8} = 3435.2\times10^{-8}\,\text{m}^4,$$

$$I_{y_c}^{\neg} = I_{y_2}^{\neg} + a_2^2 A_2 = 774\times10^{-8} + 6.585^2\times31.4\times10^{-8} = 2135.6\times10^{-8}\,\text{m}^4,$$

$$I_{y_c} = (3435.2 + 2135.6)10^{-8} = 5570.8\times10^{-8}\,\text{m}^4.$$

4. Calculation of the product of inertia relative to x_c, y_c axes.

$$I_{x_c y_c} = I_{x_c y_c}^{\vdash\dashv} + I_{x_c y_c}^{\neg},$$

For the first part of the section:

$$I_{x_c y_c}^{\vdash\dashv} = I_{x_1 y_1}^{\vdash\dashv} + a_1 c_1 A_1 = 0 + 7.715(-3.615)\times10^{-4}\times26.8\times10^{-4} = -747.4\times10^{-8}\,\text{m}^4.$$

For second part the similar approach is used:

$$I_{x_c y_c}^{\neg} = I_{x_2 y_2}^{\neg} + a_2 c_2 A_2.$$

The value of $I_{x_2 y_2}^{\neg}$ should be determined beforehand using transformation equations for product of inertia and taking into account that in rotation of axes the sum of axial moments of inertia is unchanged, i.e. $I_{x_2} + I_{y_2} = I_{max} + I_{min}$. The axes rotating procedure is shown in Fig. 2. The x_3, y_3 axes are selected as reference axes in this rotation to x_2, y_2 axes. Due to cross-section symmetry relative to x_3 axis, the angle of rotation is $\theta_p = -45°$ (clockwise rotation).

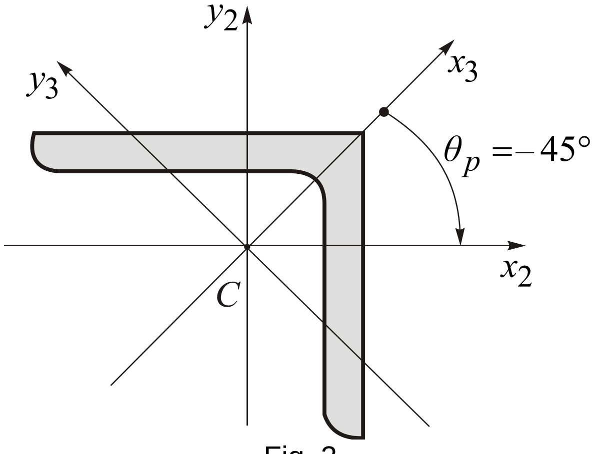


Fig. 2

In our case, general view of transformation equation for product of inertia

$$I_{x_1 y_1} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

will be rewritten as

$$I_{x_2y_2} = \frac{I_{x_3} - I_{y_3}}{2}\sin 2\theta_p + I_{x_3y_3}\cos 2\theta_p .$$

After substituting,

$$I_{x_2y_2} = \frac{1229\times10^{-8} - 319\times10^{-8}}{2}\sin(-90°) + 0\cos(-90°) = -455\times10^{-8}\text{ m}^4.$$

In our designations, this product will be denoted as $I^{\urcorner}_{x_2y_2} = -455\times10^{-8}\text{m}^4$.

Consequently,

$$I^{\urcorner}_{x_cy_c} = -455\times10^{-8} + (-6.585)(3.085)\times31.4\times10^{-8} = -1092.9\times10^{-8}\text{ m}^4 .$$

Total result after substitutions is

$$I_{x_cy_c} = (-747.4 - 1092.9)\times10^{-8} = -1840.3\times10^{-8}\text{ m}^4 .$$

5. Rotating central x_c, y_c axes to central principal position at θ_p angle.

Substituting the values of central moments and product of inertia into the equation for the angle θ_p, we get

$$\text{tg}2\theta_p = \frac{2I_{x_cy_c}}{I_{y_c} - I_{x_c}} = \frac{2\times(-1840.3)}{5570.8 - 1538} = -0.9127 \Rightarrow 2\theta_p = -42^{\circ}24' \Rightarrow \theta_p = -21^{\circ}12'.$$

Note, that this angle is clockwise due to used sign convention. It is shown in resultant picture shown below (Fig. 3).

It is important to note that in any rotation of axes to principal position larger of two axial moments of inertia ($I_{y_c} = 5570.8\,\text{cm}^4$) becomes the largest (maximum) and smaller one ($I_{y_c} = 1538\,\text{cm}^4$) becomes the minimum in value.

6. Calculation of principal central moments of inertia for composite area.

The principal moments of inertia are determined using the formula

$$I_{UV} = I_{\substack{\max\\ \min}} = \frac{I_{x_c} + I_{y_c}}{2} \pm \sqrt{\left(\frac{I_{x_c} - I_{y_c}}{2}\right)^2 + I^2_{x_cy_c}} = \left(3554.4 \pm 2293.2\right)\times10^{-8}\text{ m}^4 ,$$

$$I_U = I_{\max} = 5847.6\times10^{-8}\text{ m}^4, \quad I_V = I_{\min} = 1261.2\times10^{-8}\text{ m}^4 .$$

Note, both values must be positive!

7. Checking the results:

(a) Checking the correspondence: $I_{\max} > I_{y_c} > I_{x_c} > I_{\min}$ (in the case $I_{y_c} > I_{x_c}$)

or

$I_{\max} > I_{x_c} > I_{y_c} > I_{\min}$ (in the case $I_{x_c} > I_{y_c}$).

In our case, $5847.6\times10^{-8} > 5570.8\times10^{-8} > 1538\times10^{-8} > 1261.2\times10^{-8}$.

(b) Checking the constancy of the sum of axial moments of inertia in rotating the axes:

$$I_{\max} + I_{\min} = I_{x_c} + I_{y_c}, \rightarrow$$

$$5847.6\times10^{-8} + 1261.2\times10^{-8} = 5570.8\times10^{-8} + 1538\times10^{-8},$$

$$\left(7108.8\times10^{-8} = 7108.8\times10^{-8}\right).$$

(c) Calculating the evidently zero central principal product of inertia of the section:

$$I_{UV} = I_{x_c y_c}\cos 2\theta_p + \frac{I_{y_c} - I_{z_c}}{2}\sin 2\theta_p =$$

$$= \left[(-1804.3)\times 0.7384 + \frac{1538 - 5570.8}{2}\times\left(-0.6743\right)\right]\times10^{-8} = \left(-1358.9 + 1359\right)\times10^{-8}\ \text{m}^4 \cong 0.$$

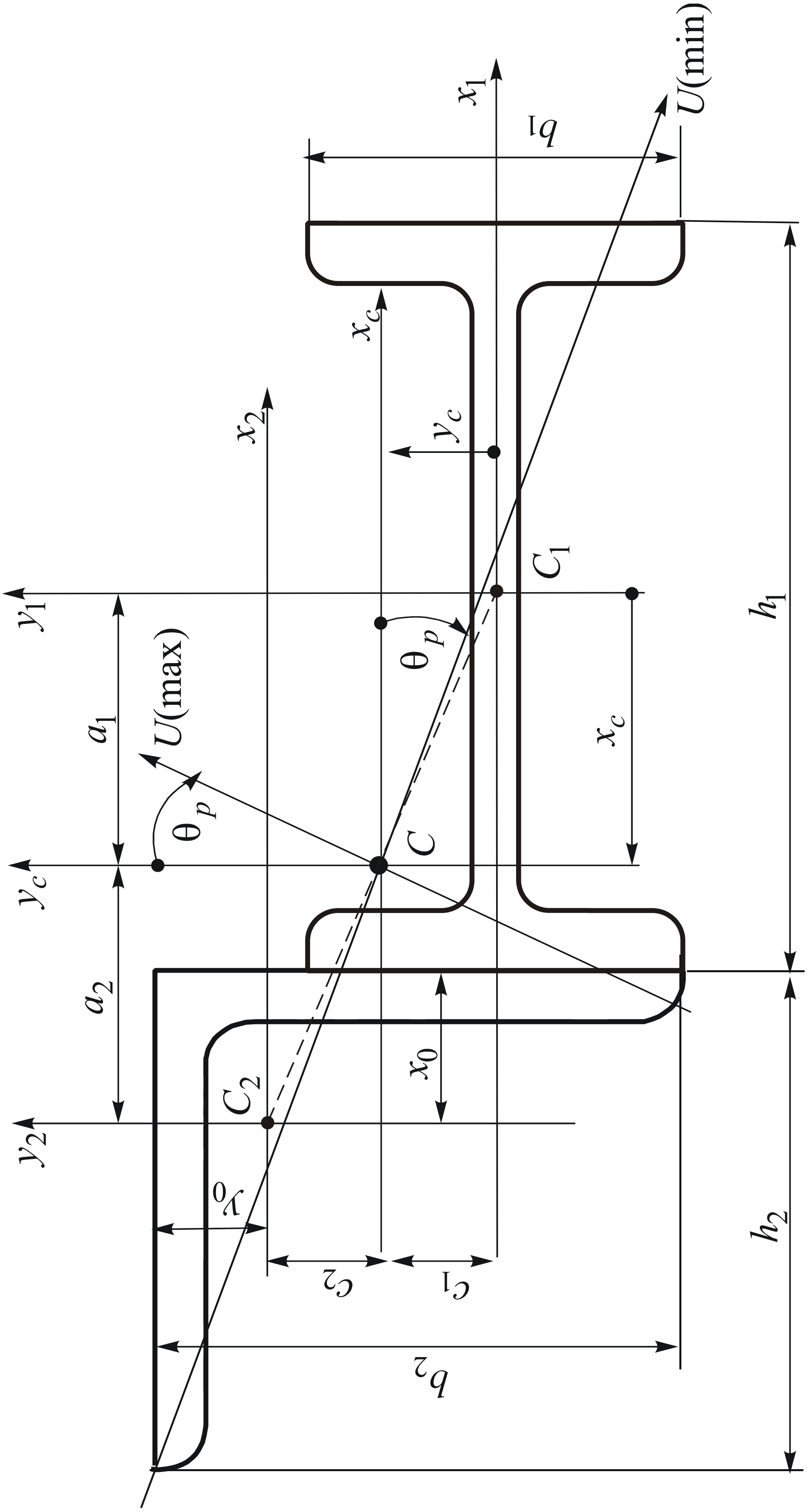

U(min)
x1
b1
xc
x2
yc
C1
y1
θp
U(max)
a1
xc
C
θp
yc
a2
x0
C2
y2
y0
c2
c1
b2
h1
h2

Fig. 3