LECTURE 6 Plane Bending Deformation. Diagrams of Internal Forces (Continued)

1 Construction of Bending Moment and Shearing Force Diagrams for Two Supported Beams

In this mode of loading, first of all, we should determine the support reactions. Let us consider the method in following example.

Example 1 Internal forces induced by concentrated force

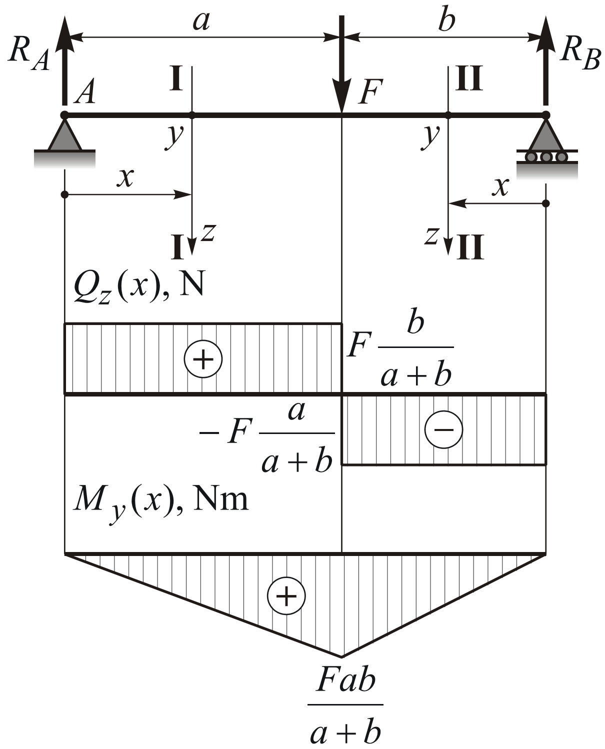


Fig. 1

Given: F, a, b.

R.D.: $Q_z(x)$, $M_y(x)$.

Let us find the support reactions R_B and R_A from the equations of equilibrium of the beam:

$$\sum M_A = 0 \rightarrow R_B(a+b) - Fa = 0, \quad (1)$$

$$R_B = \frac{Fa}{a+b}, \quad (2)$$

$$\sum M_B = 0 \rightarrow$$

$$R_A(a+b) - Fb = 0, \quad (3)$$

$$R_A = \frac{Fb}{a+b}. \quad (4)$$

Let us check the support reactions

$$\sum F_z = 0, \quad \frac{Fb}{a+b} - F + \frac{Fa}{a+b} = 0. \quad (5)$$

The result is true.

Let us divide the beam into portions and write the equations of shearing forces and bending moments for an arbitrary cross-sections of two portions: I-I and II-II. From the conditions of equilibrium of the left (I-I) or right (II-II) portion of the mentally cut rod it follows that **for portion I at a distance *x* from support *A* internal forces are described by the equations**.

$$Q_z^I(x) = R_A = \frac{Fb}{a+b}, \qquad M_y^I(x) = R_A x = \frac{Fb}{a+b} x\Big|_{x=0} = 0\Big|_{x=a} = \frac{Fba}{a+b}. \qquad (6)$$

Portion II. An arbitrary section II is situated at a distance *x* within the limits

$$0 \le x \le b: \qquad Q_z^{II}(x) = -R_B = -\frac{Fa}{a+b}, \qquad (7)$$

$$M_y^{II}(x) = R_B x = \frac{Fa}{a+b} x\Big|_{x=0} = 0\Big|_{x=b} = \frac{Fab}{a+b}. \qquad (8)$$

Thus, in each portion of the beam, the force Q_z is constant in value and is positive for portion I and negative for portion II.

The moment depends linearly on *x*; it increases in portion I from 0 to $Fab/(a+b)$ and increases from zero to this value in portion II. The diagrams of $Q_z(x)$ and $M_y(x)$ can now be created using results of these calculations.

It should be noted that the diagram of shearing forces has an abrupt in the point of application of specified external force, which is equal to the magnitude of the force.

For instance, the $Q_z(x)$ diagram has three abrupts corresponding to the forces R_A, F, R_B. There are no any abrupt in the diagram of bending moments.

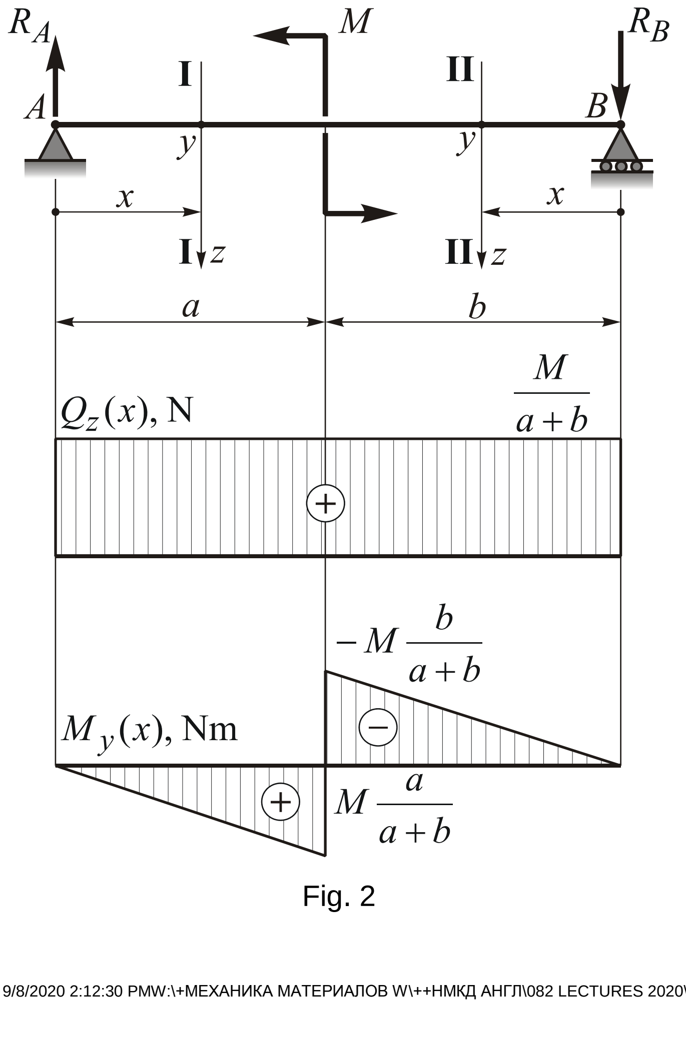


Fig. 2

Example 2 Internal forces induced by concentrated moment

Given: *M*, *a*, *b*.

R.D.: $Q_z(x)$, $M_y(x)$.

From the equations of equilibrium, the reactions are

$$R_A = R_B = \frac{M}{a+b}, \qquad (9)$$

since

$$\sum M_A = 0; \quad R_B(a+b) - M = 0,$$

$$R_B = \frac{M}{a+b}, \; R_B = R_A. \qquad (10)$$

Equations of internal forces:

Portion I: $0 \le x \le a$

$$Q_z^I(x) = R_A = \frac{M}{a+b}, \qquad (11)$$

$$M_y^I(x) = R_A x = \frac{Mx}{a+b}\bigg|_{x=0} = 0\big|_{x=a} = \frac{Ma}{a+b}. \quad (12)$$

Portion II: $0 \le x \le b$

$$Q_z^{II}(x) = +R_B = +\frac{M}{a+b}, \quad (13)$$

$$M_y^{II}(x) = -R_B x = -\frac{Mx}{a+b}\bigg|_{x=0} = 0\big|_{x=b} = -\frac{Mb}{a+b} \quad (14)$$

Note: ***point of an external moment application, the bending moment diagram has "jump" (abrupt). It is equal to the magnitude of the active moment applied in this point****.*

Example 3 Internal forces induced by uniformly distributed load

Given: q, l.

R.D.: $Q_z(x)$, $M_y(x)$.

Let us determine the reactions of supports A and B from conditions of equilibrium:

$$R_A = R_B = \frac{ql}{2}. \quad (15)$$

Equations of internal forces:

In portion I $(0 \le x \le l)$:

$$Q_z^I(x) = \frac{ql}{2} - qx\big|_{x=0} = \frac{ql}{2}\bigg|_{x=l} = -\frac{ql}{2}, \quad (16)$$

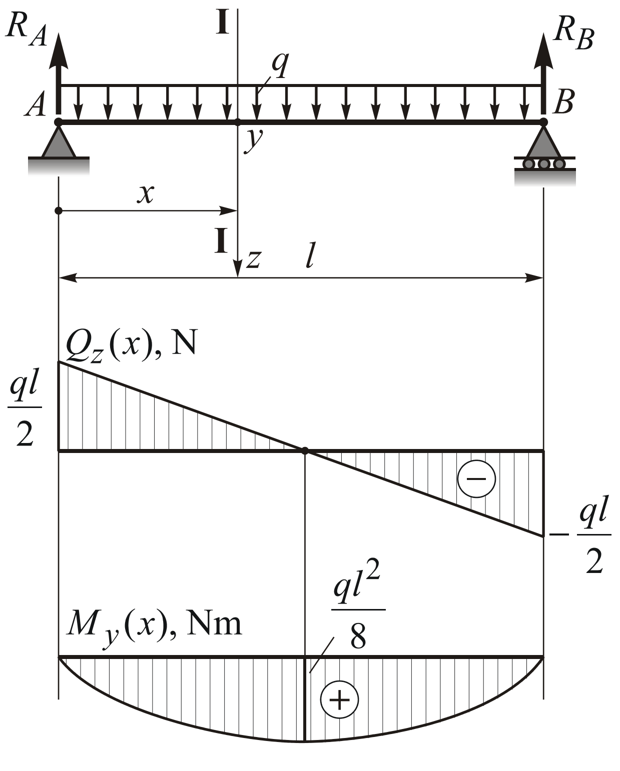


Fig. 3

$$M_y^I(x) = \frac{ql}{2}x - \frac{qx^2}{2}\bigg|_{x=0} = 0\big|_{x=l/2} = \frac{ql^2}{8}\bigg|_{x=l} = 0. \quad (17)$$

The shearing force decreases linearly from $\frac{ql}{2}$ to $-\frac{ql}{2}$ and the bending moment varies along the beam length non-linearly and attains the maximum value in the beam mid-section.

Example 4 Internal forces induced by linearly distributed load

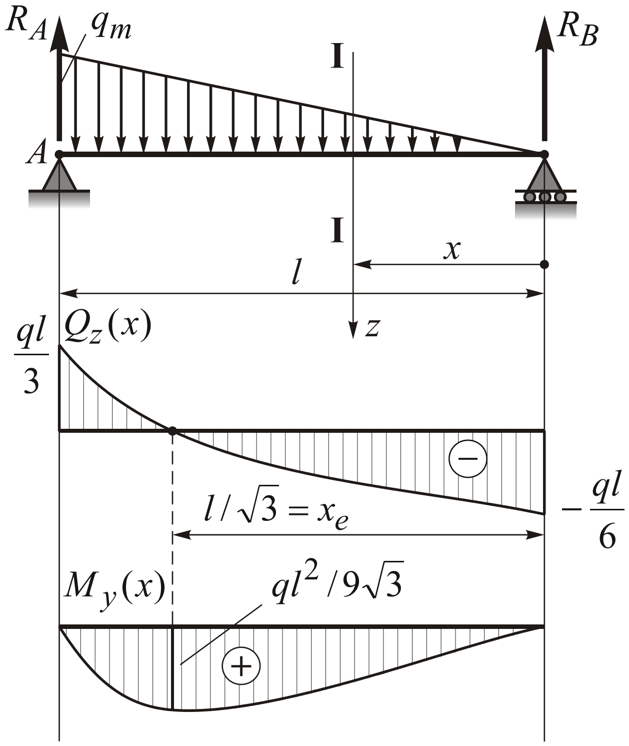


Fig. 4

Given: q_m, l.

R.D.: $Q_z(x)$, $M_y(x)$.

Calculation of the support reactions:

$$\sum M_A = 0,\ R_B l - \frac{ql}{2}\frac{1}{3}l = 0, \quad (18)$$

$$R_B = \frac{ql}{6}, \quad (19)$$

$$\sum M_B = 0,\ R_A l - \frac{ql}{2}\frac{2}{3}l = 0, \quad (20)$$

$$R_A = \frac{ql}{3}, \quad (21)$$

Checking:

$$\sum F_z = 0, \quad \frac{ql}{6} + \frac{ql}{3} - \frac{ql}{2} = 0. \quad (22)$$

The equations of $Q_z(x)$ and $M_y(x)$ are as follows:

$$Q_z^I(x) = -R_B + \frac{qx^2}{2l} = -\frac{ql}{6} + \frac{qx^2}{2l}\bigg|_{x=0} = -\frac{ql}{6}\bigg|_{x=l} = \frac{ql}{3}; \quad (23)$$

Extreme value of bending moment is calculated substituting into its equation corresponding coordinate of the point. The last one is determined equating to zero the derivative of bending moment function which is described by shear force (check if you disagree):

$$Q_z^I(x_e) = -\frac{ql}{6} + \frac{q{x_e}^2}{2l} = 0 \rightarrow x_e = \frac{l}{\sqrt{3}}. \quad (24)$$

$$M_y^I(x) = R_B x - \frac{qx^3}{6l} = \frac{ql}{6}x - \frac{qx^3}{6l}\bigg|_{x=0} = 0\big|_{x=l} = 0\big|_{x=x_e} = \frac{ql^2}{9\sqrt{3}}. \quad (25)$$

2 Relationships Between Bending Moment M_y, Shearing Force Q_z and Distributed Load Intensity $q(x)$

Let a rod be fixed in arbitrary manner and subjected to a distributed load of intensity $q = q(x)$ in general case. There are other loads applied to the beam.

The direction adopted for *q* is considered positive (see Fig. 5).

Isolate an element of length *dx* from the rod and apply to both its boundaries the moments M_y and $M_y + dM_y$ and also shearing forces Q_z and $Q_z + dQ_z$.

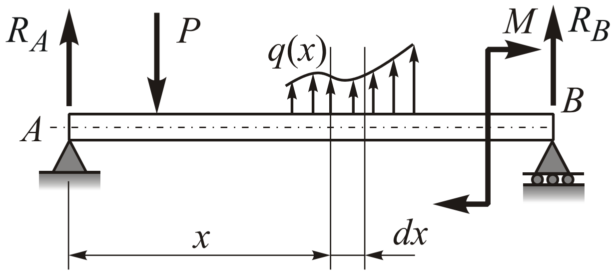


Fig. 5

The load *q* may be considered to be uniformly distributed within the infinitesimally small segment *dx* (see Fig. 6).

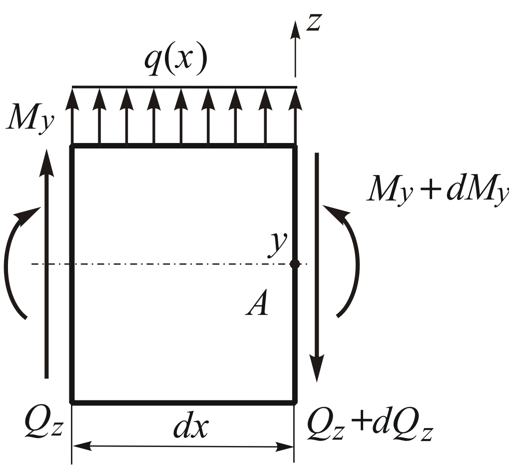


Fig. 6

Equate to zero the sum of the projections of all forces on a vertical axis *z*:

$$Q_z + q(x)dx - Q_z - dQ_z = 0. \quad (26)$$

Making simplifications, we obtain

$$\frac{dQ_z(x)}{dx} = q(x). \quad (27)$$

Equate to zero the sum of the moments with respect a transverse axis *y* (point *A*)

$$M_y + Q_z \cdot dx + q(x)dx\frac{dx}{2} - M_y - dM_y = 0. \quad (28)$$

Making simplifications and rejecting the small quantity of higher order, we obtain

$$\frac{dM_y(x)}{dx} = Q_z(x). \tag{29}$$

Thus the shearing force in fact represents the derivative of the bending moment with respect to the length of the rod. This conclusion was proved in the example mentioned above.

The derivative of the shearing force equals to the intensity of external distributed load q.

Example 5 General case of cantilever beam loading

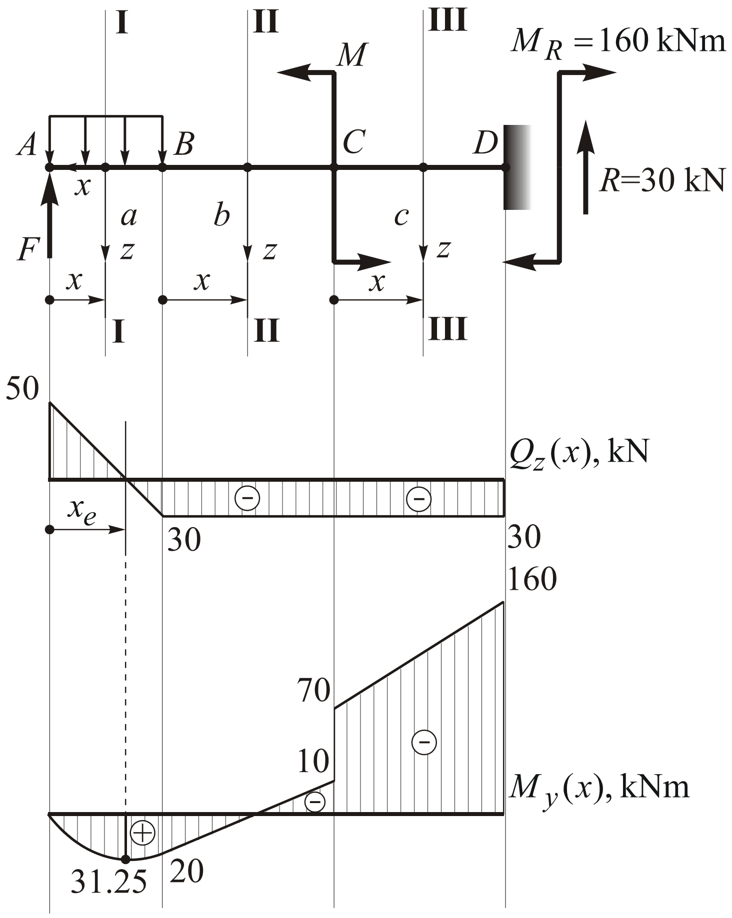


Fig. 7

Given: $a = 2\,\text{m}$, $b = 1\,\text{m}$, $c = 3\,\text{m}$, $F = 50\,\text{kN}$, $q = 40\,\text{kNm}$, $M = 60\,\text{kNm}$.

R.D.: $Q_z(x)$, $M_y(x) - ?$

Equations of internal forces in an arbitrary cross-sections of the portions:

I–I $0 < x < a$

$$Q_z^I(x) = +F - qx\big|_{x=0} = 50\big|_{x=2} =$$

$$= -30\,\text{kN},$$

$$M_y{}^I(x) = +Fx - qx\frac{x}{2}\Big|_{x=0} =$$

$$= 0\big|_{x=2} = 20\,\text{kNm}.$$

$$M_{y\max}^I - ?$$

$$\left|Q_z{}^{\text{I}}\right| = \frac{dM}{dx} = F - qx; \qquad Q_z(x_e) = 0.$$

$$\frac{dM(x_e)}{dx} = F - qx_e = 0 \rightarrow x_e = \frac{F}{q} = 1.25\,\text{m}.$$

$$M^{I}_{y_{\max}} = M^{I}_{y}(x_e) = F(x_e) - \frac{q(x_e)^2}{2} = 50(1.25) - \frac{40(1.25)^2}{2} = 62.5 - 31.25 = 31.25\,\text{kNm}.$$

II–II $0 < x < b$

$$Q_z^{II}(x) = +F - qa = -30\,\text{kN},$$

$$M_y^{II}(x) = F(a+x) - qa\left(\frac{a}{2}+x\right)\Big|_{x=0} = 50\times(2+0) - 40\times 2(1) = 20\Big|_{x=1} = -10\,\text{kNm}.$$

III–III $0 < x < b$

$$Q_z^{III}(x) = F - qa = -30\,\text{kN},$$

$$M_y^{III}(x) = F(a+b+x) - qa\left(\frac{a}{2}+b+x\right) - M\Big|_{x=0} = -10\Big|_{x=1} = -70\,\text{kNm}.$$

Example 6 General case of two supported beam loading

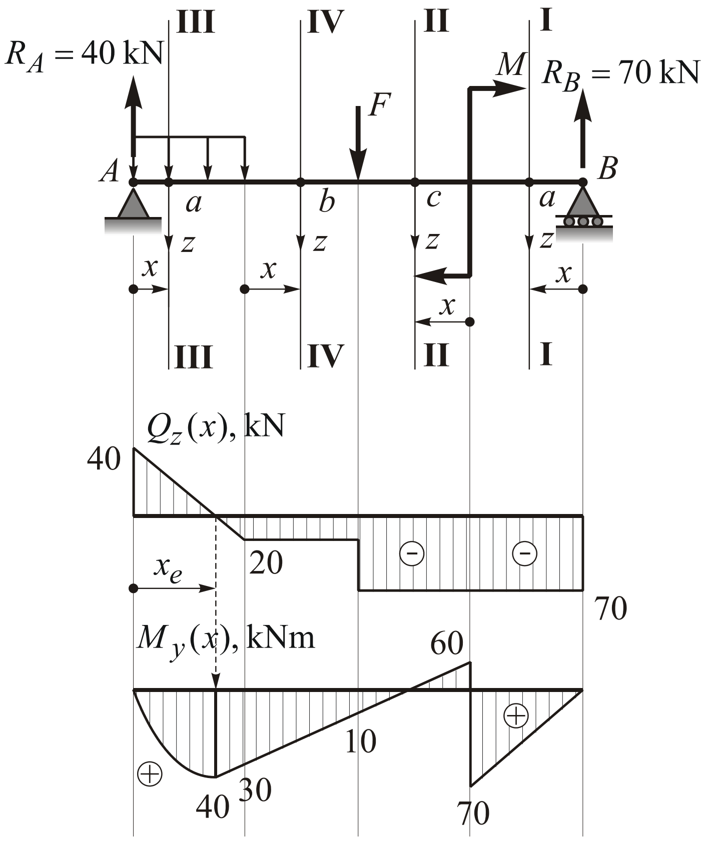


Fig. 8

Given: $q = 20\,\text{kNm}$, $F = 50\,\text{kN}$, $M = 130\,\text{kNm}$, $a = 3\,\text{m}$, $b = c = d = 1\,\text{m}$.

R.D.: $Q_z(x)$, $M_y(x) - ?$

1) Calculation of reactions R_A, R_B:

$$\sum M_A = 0 = +M + F(a+b) +$$

$$+qa\frac{a}{2} - R_B(a+b+c+d),$$

$$R_B = +70\,\text{kN}.$$

The plus sign means coincidence of R_B actual direction with originally selected.

$$\sum M_B = 0 = +M - F(c+d) - qa\left(\frac{a}{2} + b + c + d\right) + R_A(a+b+c+d),$$

$$R_A = +40\,\text{kN}.$$

Checking: $\sum F_z = 0 = -R_A - R_B + F + qa = -40 - 70 + 50 + 20 \times 3$, i.e. the reactions are determined correctly.

2) Equations of internal forces in an arbitrary cross-sections of corresponding portions:

I–I $0 < x < 1$

$$Q_z^I(x) = -R_B = -70\,\text{kN},$$

$$M_y{}^I(x) = +R_B x\big|_{x=0} = 0\big|_{x=1} = 70\,\text{kNm}.$$

II–II $0 < x < 1$

$$Q_z^{II}(x) = -R_B = -70\,\text{kN},$$

$$M_y{}^{II}(x) = +R_B(d+x) - M\big|_{x-0} = 60\big|_{x=1} = 10\,\text{kNm}.$$

III–III $0 < x < 1$

$$Q_z^{III}(x) = R_A - qx\big|_{x=0} = 40\big|_{x=3} = -20\,\text{kNm},$$

$$M_{y\max}^{III} - ?$$

$$Q_z^{III}(x_e) = R_A - qx_e = 0 \rightarrow x_e = \frac{R_A}{q} = \frac{40}{20} = 2\,\text{m},$$

$$M^{III}_{y}(x_e) = R_A x_e - q\frac{x_e{}^2}{2} = +40\,\text{kNm}.$$

IV–IV $0 < x < 1$

$$Q_z^{IV}(x) = R_A - qa = -20\,\text{kN},$$

$$M_y{}^{IV}(x) = +R_A(a+x) - qa\left(\frac{a}{2} + x\right)\bigg|_{x=0} = 30\big|_{x=1} = 10\,\text{kNm}.$$

Example 7 General case of two supported beam loading

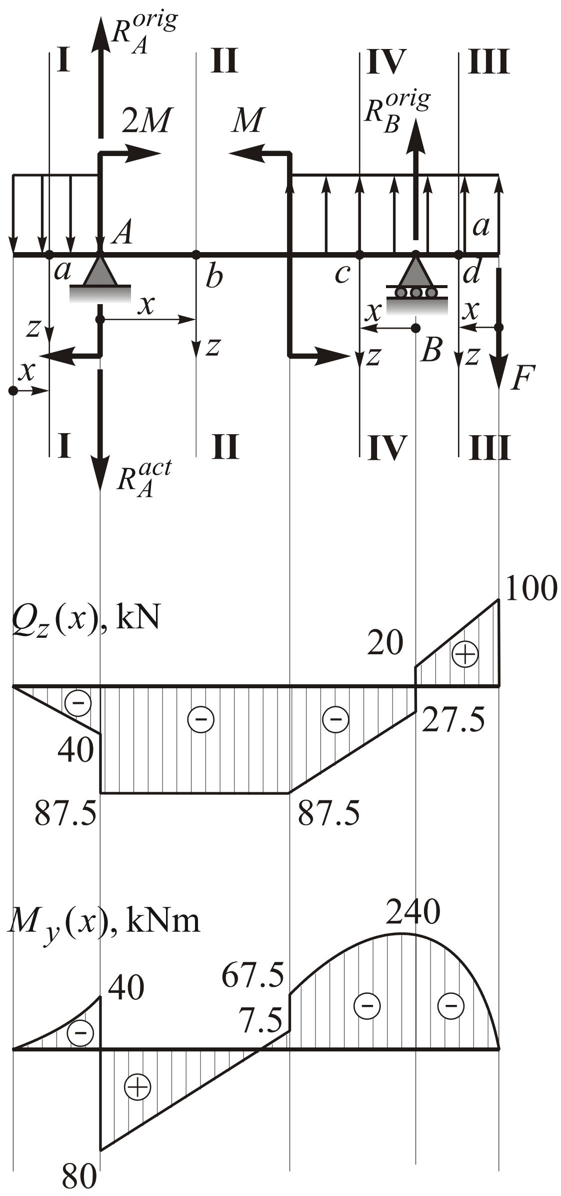


Fig. 9

Given: $q = 20\,\text{kN/m}$, $F = 100\,\text{kN}$, $M = 60\,\text{kNm}$, $a = 2\,\text{m}$, $b = 1\,\text{m}$, $c = 3\,\text{m}$, $d = 4\,\text{m}$.

R.D.: $Q_z(x)$, $M_y(x) - ?$

1) Calculation of reactions R_A, R_B:

$$\sum M_A = 0 = -R_B^{orig}(b+c) + F\,(b+c+d) -$$

$$-q(c+d)\left(\frac{c+d}{2}+b\right) - M + 2M - qa\frac{a}{2} =$$

$$= -R_B(4) + 100(8) - 20(7)\left(\frac{7}{2}+1\right) -$$

$$-60 + 120 - 20(2), \quad -R_B 4 = -190,$$

$$R_B^{orig} = R_B^{act} = \frac{190}{4} = +47.5\,\text{kN}.$$

The plus sign means coincidence of R_B actual direction with originally selected.

(b) $\sum M_B = 0 = +Fd + 2M - M +$

$$+qc\frac{c}{2} - qd\frac{d}{2} + R_A^{orig}(b+c) - qa\left(\frac{a}{2}+b+c\right) =$$

$$= 100(4) + 2(60) - 60 + 20\frac{9}{2} -$$

$$-20\frac{16}{2} + R_A(4) - 40(1+1+3) = R_A^{orig}(4) + 190, \; R_A^{orig} = -47.5\,\text{kN}.$$

The minus sign means non-coincidence of R_B actual direction with originally selected. Thus, original direction of R_A must be changed on opposite.

(c) Checking: $\sum F_z = 0 = +R_A^{act} - R_B^{orig} + F + qa - q(c+d) = 0$, i.e. the reactions are determined correctly.

2) Equations of internal forces in an arbitrary cross-sections of corresponding portions:

I–I $0 < x < a$

$$Q_z^I(x) = -qx\Big|_{x=0} = 0\Big|_{x=2} = -40\,\text{kN}, \qquad M_y^I(x) = qx\frac{x}{2}\Big|_{x=0} = 0\Big|_{x=2} = -40\,\text{kNm}.$$

II–II $0 < x < b$

$$Q_z^{II}(x) = -qa - R_{Aactual} = -87.5\,\text{kN},$$

$$M_y^{II}(x) = -qa\left(\frac{a}{2} + x\right) - R_{Aactual}x + 2M\Big|_{x=0} = +80\Big|_{x=1} = -7.5\,\text{kNm}.$$

III–III $0 < x < d$

$$Q_z^{III}(x) = +F - qx\Big|_{x=0} = +100kN\Big|_{x=4} = +20\,\text{kN},$$

$$M_y^{III}(x) = -Fx + qx\frac{x}{2}\Big|_{x=0} = 0\Big|_{x=4} = -240\,\text{kNm}.$$

IV–IV $0 < x < c$

$$Q_z^{IV}(x) = +F - q(d + x) - R_B\Big|_{x=0} = -27.5kN\Big|_{x=3} = -87.5\,\text{kN},$$

$$M_y^{IV}(x) = -F(d + x) + R_Bx + q(d + x)\times\left(d + \frac{x}{2}\right)\Big|_{x=0} = -240kN\Big|_{x=3} = -67.5\,\text{kNm}.$$

3 Construction of Internal Force Diagrams for Statically Determinated Frames

By a **bar system** is meant any structure consisting of rod-shaped elements. If the elements of a structure act primarily in tension or compression, the bar system is called a **truss**. If the elements of a bar system are primarily in bending or torsion, the system is called a **frame**. We will consider plane frames.

By a **statically determinate system** is meant a system for which all the reactions of the supports can be determined by means of equations of equilibrium and the internal forces at any cross section can also be found by the method of sections.

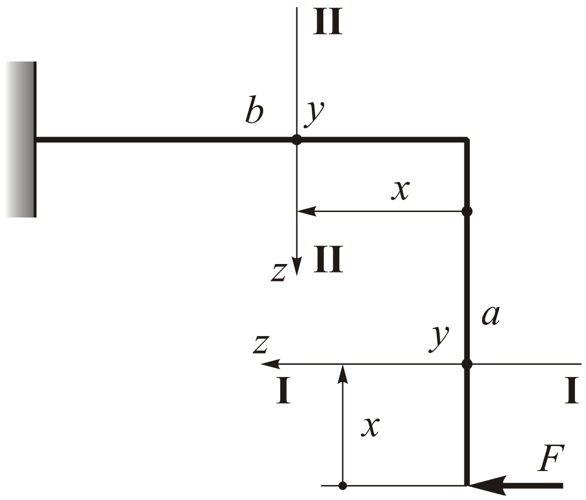


Fig. 10

Given: F, a, b.

R.D.: $N_x(x)$, $Q_z(x)$, $M_y(x)$ functions and their graphs.

Solution

Equations of internal forces in two portions are:

I–I $0<x<a$

$$N^I(x)=0, \qquad Q_z^I(x)=0,$$

$$M_y^I(x)=-Fx\big|_{x=0}=0\big|_{x=a}=-Fa,$$

II–II $0<x<b$

$$N^{II}(x)=-F, \qquad Q_z^{II}(x)=0, \qquad M_y^{II}(x)=-Fa.$$

Corresponding graphs are:

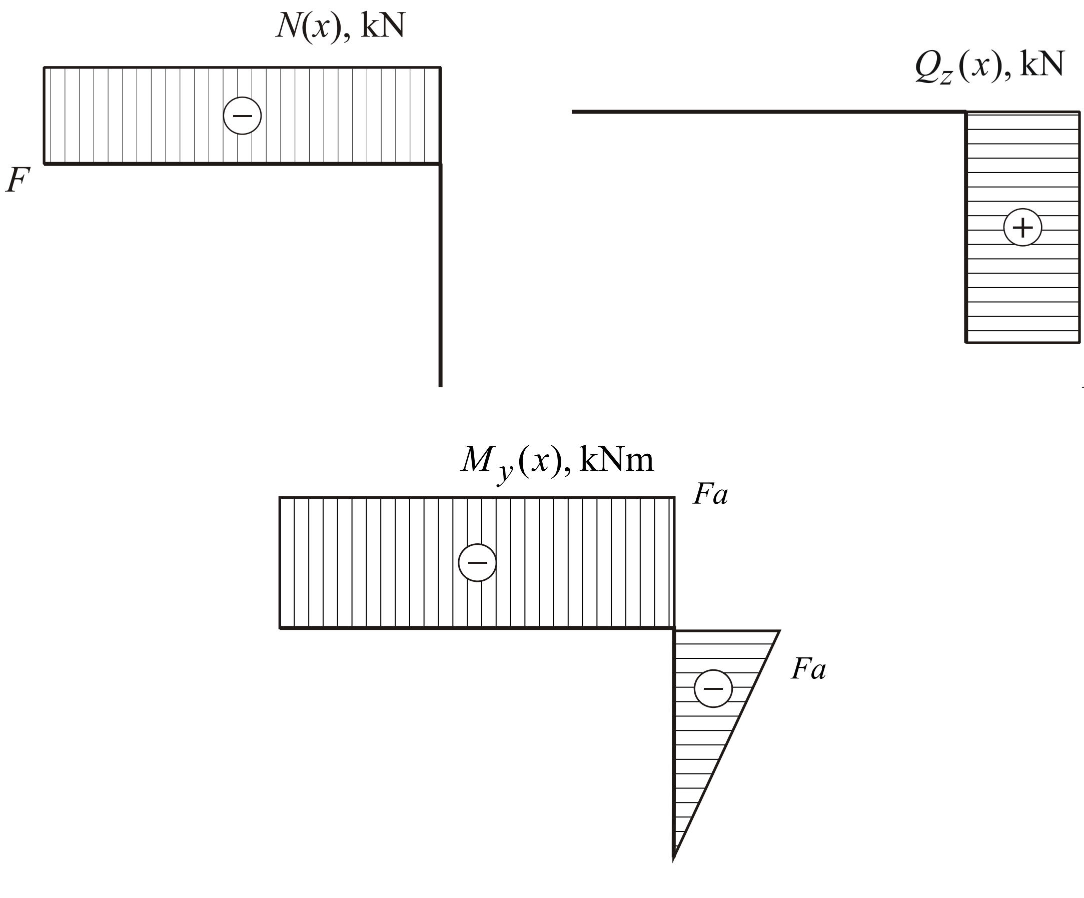


Fig. 11

Example 9 Calculation of internal forces in plane frame

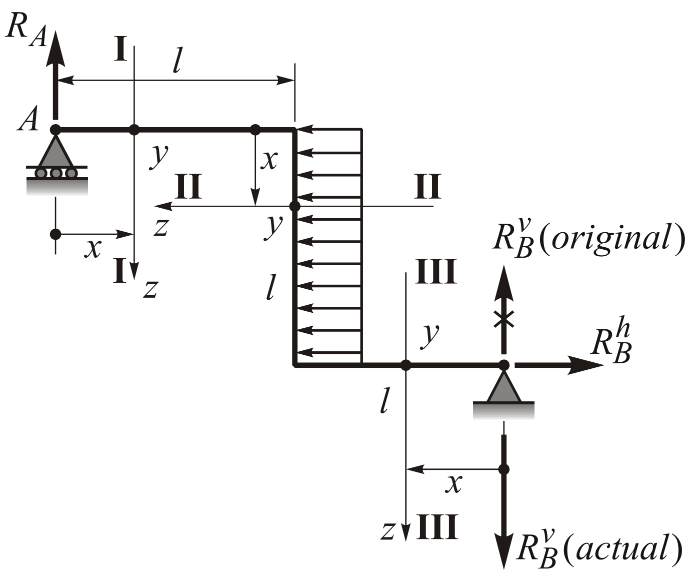


Fig. 12

Given: $q = 20\,\text{kN/m}$, $l = 3\,\text{m}$.

R.D.: $N_x(x)$, $Q_z(x)$, $M_y(x)$.

At first let us determine the reactions in supports A and B:

a) $\sum M_B = 0$,

$R_A \cdot 6 - 20\times 3\times 1.5 = 0$, $R_A = +15\,\text{kN}$;

b) $\sum F_x = 0$,

$R_B^h = qb = +60\,\text{kN}$:

c) $\sum M_A = 0$,

$R_{Bv}^{orig} \cdot 6 + 60\times 3 - 20\times 3\times 1.5 = 0$,

$R_{Bv}^{orig} = -15\,\text{kN}$ (actual direction of R_{Bv}^{act} is opposite to original).

Let us write the equations of the internal forces:

$N^I{}_x(x) = 0$;

$N^{II}{}_x(x) = +R_A = +15\,\text{kN}$;

$N^{III}{}_x(x) = +R_{Bh} = +60\,\text{kN}$;

$Q_z^I(x) = R_A = 15\,\text{kN}$;

$Q_z^{II}(x) = -qx\big|_{x=0} = 0\big|_{x=l} = -60\,\text{kN}$;

$Q_z^{III}(x) = +R_{Bv}^{act} = 15\,\text{kN}$;

$M_y^I(x) = R_A \cdot x\big|_{x=0} = 0\big|_{x=l} = 45\,\text{kNm}$;

$$M_y^{II}(x) = R_A \cdot l - \frac{qx^2}{2}\bigg|_{x=0} = +45\big|_{x=3} = -45\,\text{kNm};$$

$M_y^{III}(x) = -R_{Bv}^{act} \cdot x\big|_{x=0} = 0\big|_{x=3} = -45\,\text{kNm}$.

Using this equations the bending moment diagram and the diagrams of normal and shearing forces may be constructed:

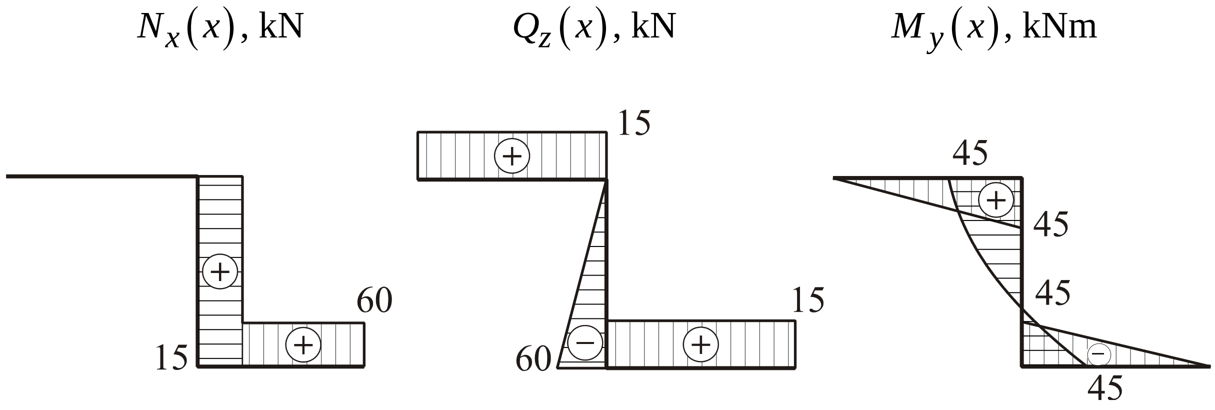


Fig. 15

Checking the results, i.e. the equilibrium of the frame angles.

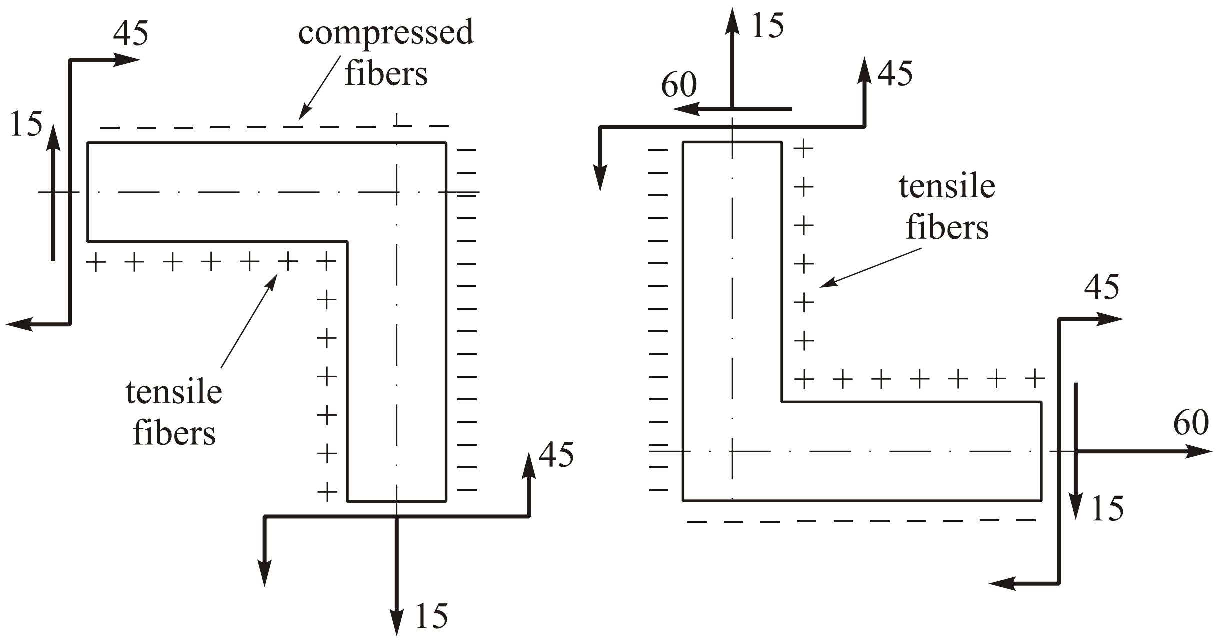


Fig. 16

Example 10 Calculation of internal forces in plane frame

Given: $q = 40\,\text{kN/m},\ F = 50\,\text{kN},\ M = 40\,\text{kNm},\ a = 2\,\text{m},\ b = 4\,\text{m},\ c = 3\,\text{m}.$

R.D.: $N_x(x),\ Q_z(x),\ M_y(x) - ?$

1) Calculation of support reactions:

(a) $\sum M_A = 0 = -M - M + Fa - qa\frac{a}{2} + qb\frac{b}{2} + R_{Bv}^{orig} b \rightarrow R_{Bv}^{orig} = -65\,\text{kN}.$ The minus sign means non-coincidence of R_{Bv} actual direction with originally selected. Thus, original direction of R_{Bv} must be changed on opposite.

(b) $\sum F_z = 0 = -F + R_A^{orig} - R_{Bv}^{act} + q(a+b) \rightarrow R_A = -125\,\text{kN}$, i.e. actual direction of R_A is upwards.

(c) $\sum F_x = 0 = R_{Bh} - F \rightarrow R_{Bh} = +50\,\text{kN}$ (right directed).

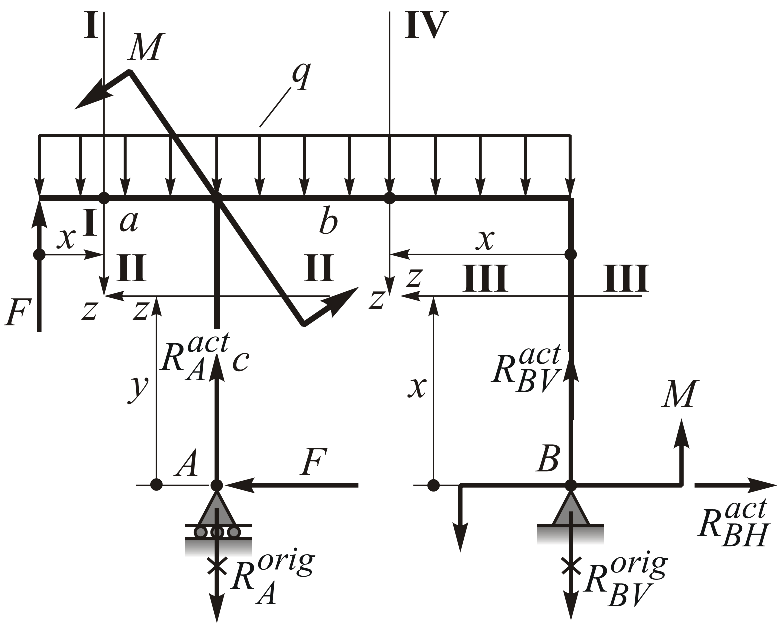


Fig. 17

2) Equations of internal forces in an arbitrary cross-sections of the portions (see Fig. 17):

I–I $0 < x < a$

$N_x^I(x) = 0$,

$Q_z^I(x) = +F - qx\big|_{x=0} =$

$= 50\big|_{x=2} = -30\,\text{kN}$,

$M_y^I(x) = Fx - qx\dfrac{x}{2}\Big|_{x=0} =$

$= 0\big|_{x=2} = 100 - 80 = +20\,\text{kNm}$,

$Q_z^I(x_e) = F - qx_e = 0$,

$$x_e = \frac{F}{q} = \frac{50}{40} = 1.25\,\text{m}.$$

$$M_y^I(x_e) = M_{y\max}^I = Fx_e - \frac{1}{2}qx_e^{\,2} = +31.25\,\text{kNm}.$$

II–II $0 < x < c$

$N_x^{II}(x) = -R_A = -125\,\text{kN}$, $\quad Q_z^{II}(x) = +F = +50\,\text{kN}$,

$M_y^{II}(x) = -Fx\big|_{x=0} = 0\big|_{x=3} = -150\,\text{kNm}$.

III–III $0 < x < c$

$N_x^{III}(x) = -R_{Bv} = -65\,\text{kN}$, $\quad Q_z^{III}(x) = -R_{Bh} = -50\,\text{kN}$,

$M_y^{III}(x) = +R_{Bh}x + M\big|_{x=0} = 40\big|_{x=3} = +190\,\text{kNm}$.

IV–IV $0 < x < c$

$$N_x^{IV}(x) = +R_{Bh} = +50\,\text{kN}, \qquad Q_z^{IV}(x) = -R_{Bv} + qx\big|_{x=0} = -65\big|_{x=4} = 95\,\text{kN},$$

$$M_y^{IV}(x) = -qx\frac{x}{2} + R_{Bv}x + M + R_{Bh}c\big|_{x=0} = 190\big|_{x=4} = 130\,\text{kNm}.$$

$$Q_z^{IV}(x_e) = -R_{Bv}^{act} + qx_e = 0 \rightarrow x_e = \frac{R_{Bv}^{act}}{q} = \frac{65}{40} = 1.625\,\text{m},$$

$$M_y^{IV}(x_e) = M_{y\max}^{IV} = -q\frac{x_e^{\,2}}{2} + R_{Bv}^{act}x_e + M + R_{Bh}^{act}c = -10\frac{(1.625)^2}{2} + 65(1.625) + 40 +$$

$$+50 \times 3 = 282.4\,\text{kNm}.$$

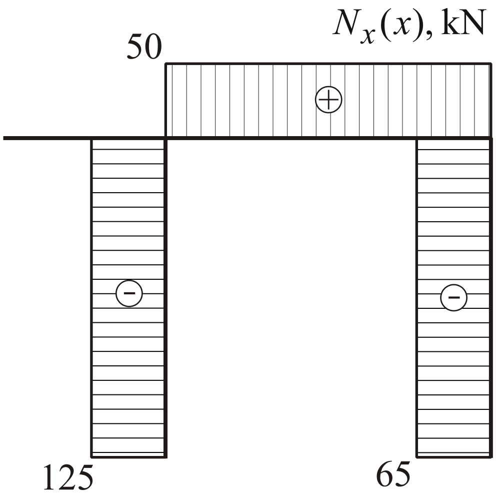


Fig. 18

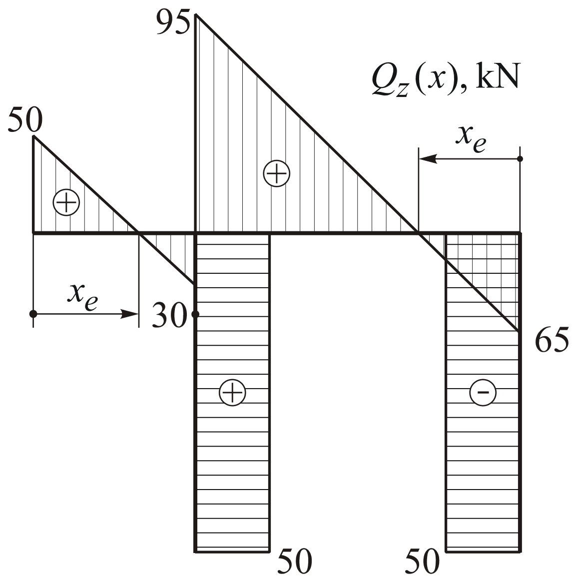


Fig. 19

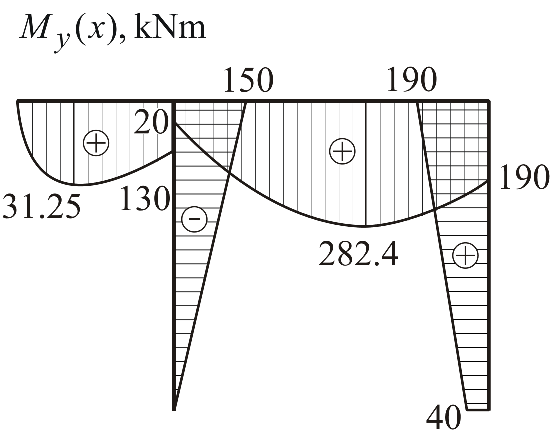


Fig. 20

3) Checking the results i.e. an equilibrium of the rods connection areas (see Figs 21–22).

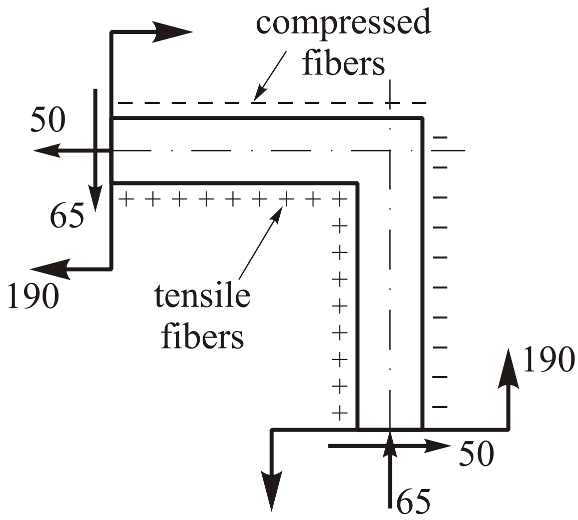


Fig. 21

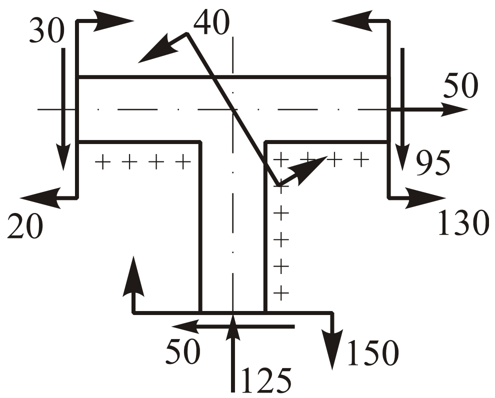


Fig. 22

4 Construction of the $Q_z(x)$ and $M_y(x)$ Diagrams for Curvilinear Beams

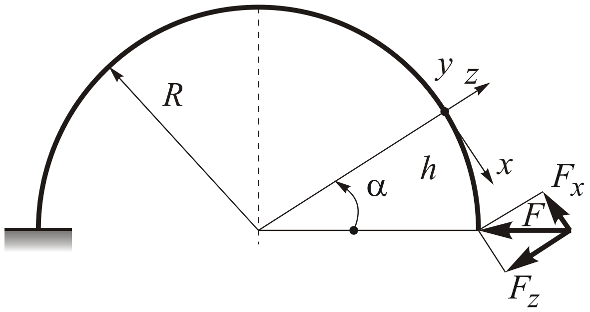


Fig. 23

Given: *R*, *F*.

R.D: $N_x(a)$, $Q_z(a)$, $M_y(a)$.

The force *F* can be resolved along the *x* and *z* axes into the components $F\cos\alpha$ and $F\sin\alpha$.

$$F_x = F\sin\alpha\,, \qquad F_z = F\cos\alpha\,,$$

$$N_x(\alpha) = -F_x = -F\sin\alpha\big|_{\alpha=0} = 0\big|_{\alpha=\pi/2} = -F\big|_{\alpha=\pi} = 0,$$

$$Q_z(\alpha) = +F_z = F\cos\ \alpha\big|_{\alpha=0} = F\big|_{\alpha=\pi/2} = 0\big|_{\alpha=\pi} = -F,$$

$$M_y(\alpha) = Fh = F\,R\sin\alpha\big|_{\alpha=0} = 0\big|_{\alpha=\pi/2} = FR\big|_{\alpha=\pi} = 0\,.$$

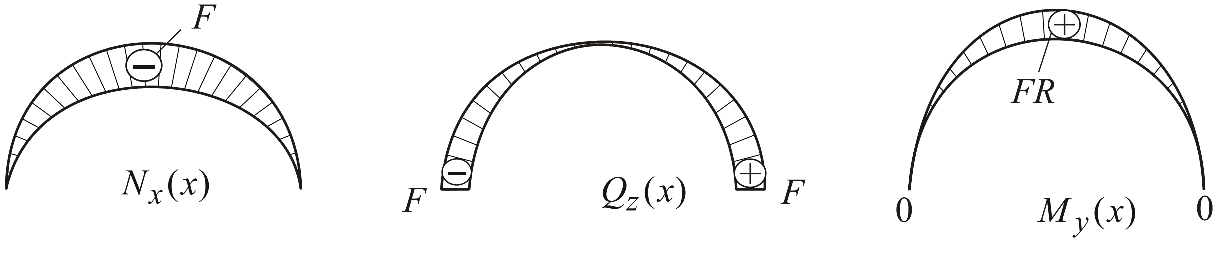


Fig. 24

Example 11 Calculation of internal forces in curvilinear frame

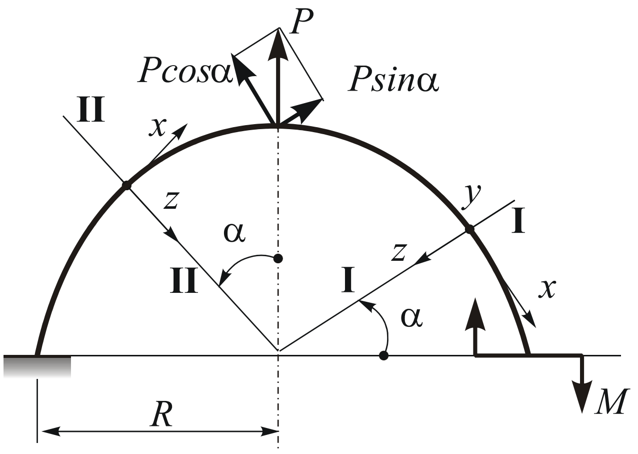


Fig. 25

Given: *M, F, R.*

R.D.: $N_x(a)$, $Q_z(a)$, $M_y(a)$.

Solution

Equations of internal forces are:

I–I $0<\alpha<\pi/2$

$N^I(\alpha)=0$; $Q_z^I(\alpha)=0$;

$M_y^I(\alpha)=-M$.

II–II $0<\alpha<\pi/2$

$$N^{II}(\alpha)=P\sin\alpha\big|_{\alpha=0}=0\big|_{\alpha=\pi/2}=P,$$

$$Q_z^{II}(\alpha)=-P\cos\alpha\big|_{\alpha=0}=-P\big|_{\alpha=\pi/2}=0,$$

$$M_y^{II}(\alpha)=-M+PR\sin\alpha\big|_{\alpha=0}=-M\big|_{\alpha=\pi/2}=-M+PR.$$

The graphs of internal forces are:

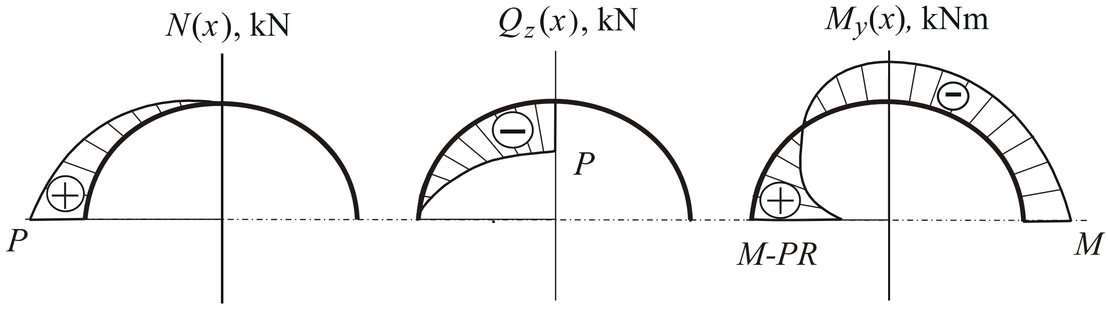


Fig. 26

Example 12 Calculation of internal forces in plane frames with curvilinear elements

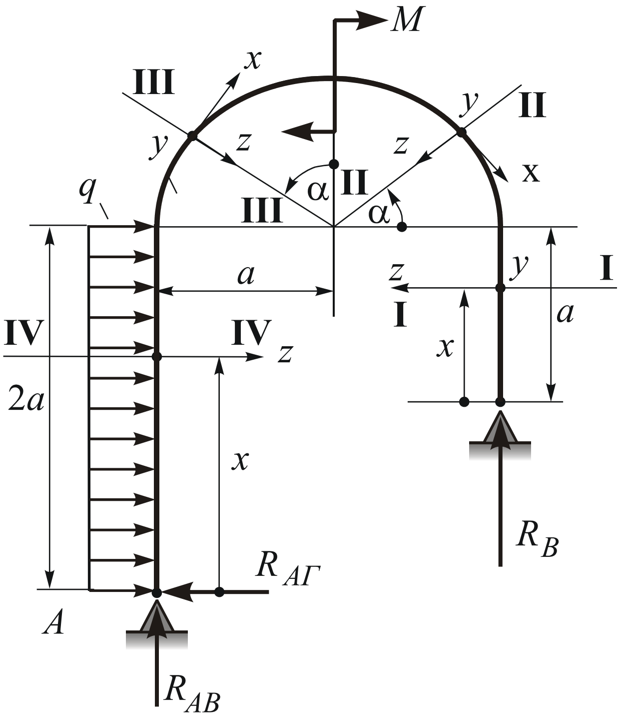


Fig. 27

Given: $M=40\,\text{kHm}$, $q=10\,\text{kN/m}$, $a=1\,\text{m}$.

R.D.: N_x, Q_z, M_y functions and their graphs.

Solution

1) Calculation of support reactions:

$$\sum F_x = 0 \quad \rightarrow \quad 2qa - R_{AH} = 0 \rightarrow$$

$$\rightarrow R_{AH} = 20\,\text{kN},$$

$$\sum M_A(F) = 0 \rightarrow R_\text{B} \cdot 2a - M - 2qa^2 = 0 \rightarrow$$

$$\rightarrow R_B = 30\,\text{kN},$$

$$\sum F_z = 0 \rightarrow R_\text{B} - R_{AV} = 0 \rightarrow R_{AV} = 30\,\text{kN}.$$

2) Equations of internal forces in an arbitrary cross-sections of corresponding portions:

I–I $0 \le x \le a$

$$N_x^I(x) = -R_B = -30\,\text{kN},\ Q_z^I(x) = 0,\ M_y^I(x) = 0.$$

II–II $0 \le \alpha \le \pi/2$

$$N_x^{II}(a) = -R_B \cos\alpha\big|_{\alpha=0} = -30\big|_{\alpha=\pi/2} = 0,$$

$$Q_z^{II}(a) = -R_B \sin\alpha\big|_{\alpha=0} = 0\big|_{\alpha=\pi/2} = -30\,\text{kN},$$

$$M_y^{II}(a) = R_B\, a(1-\cos\alpha)\big|_{\alpha=0} = 0\big|_{\alpha=\pi/2} = R_B a = 30\,\text{kNm}.$$

III–III $0 \le \alpha \le \pi/2$

$$N^{III}(a) = R_B \sin\alpha\big|_{\alpha=0} = 0\big|_{\alpha=\pi/2} = 30\,\text{kN},$$

$$Q_y^{III}(a) = -R_B \cos\alpha\big|_{\alpha=0} = -30\big|_{\alpha=\pi/2} = 0,$$

$$M_y^{III}(x) = R_B a(1+\sin\alpha) - M\big|_{\alpha=0} = -10\big|_{\alpha=\pi/2} = 20\,\text{kNm}.$$

IV–IV $0 \le \alpha \le 2a$

$$N^{IV}(x) = R_{AV} = 30\,\text{kN}, \quad Q_y^{IV}(x) = R_{AH} - qx\big|_{x=0} = 20\big|_{x=2} = 0,$$

$$M_y^{IV}(x) = R_{AH}x - \frac{qx^2}{2}\bigg|_{x=0} = 0\big|_{x=2} = 20\,\text{kNm}.$$

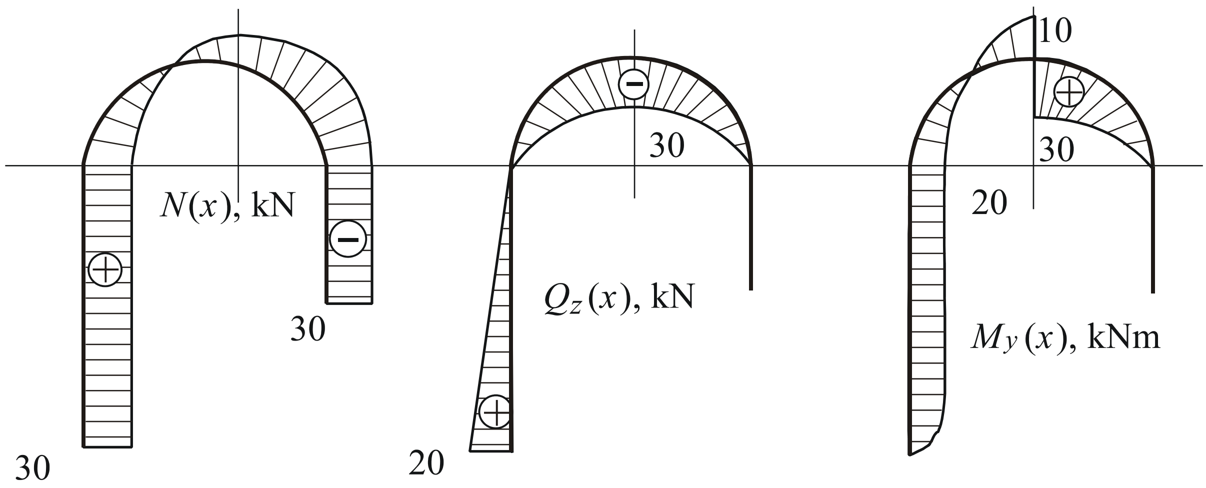


Fig. 28

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National aerospace university “Kharkiv Aviation Institute”

Department of aircraft strength

Course
Mechanics of materials and structures

HOME PROBLEM 5

Graphs of Shear Force and Bending Moment Distribution in Plane Bending (Two-Supported Beams)

Name of student:

Group:

Advisor:

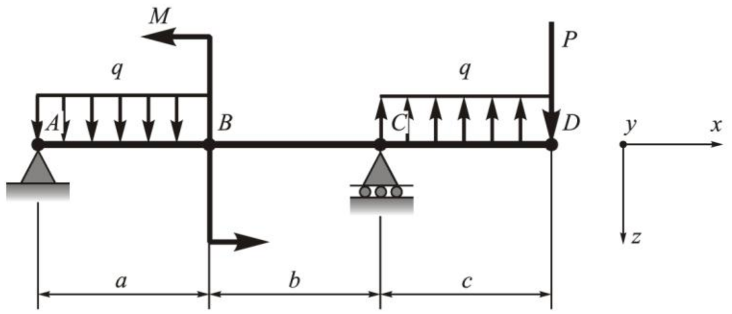
Data of submission:

Mark:

National aerospace university
"Kharkiv Aviation Institute"
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: graphs of shear force and bending moment distribution along the length of a beam in plane bending deformation.
Full name of the student, group

Variant: 1 **Complexity:** 1



Given: $q = 10$ kN/m, $M = 20$ kNm, $P = 30$ kN, $a = 2$ m, $b = 4$ m, $c = 2$ m.

Goal: obtain the equations of shear force and bending moment in the cross-sections of a beam and design the graphs of their distribution along the beam length.

Full name of the lecturer **signature**

Mark:

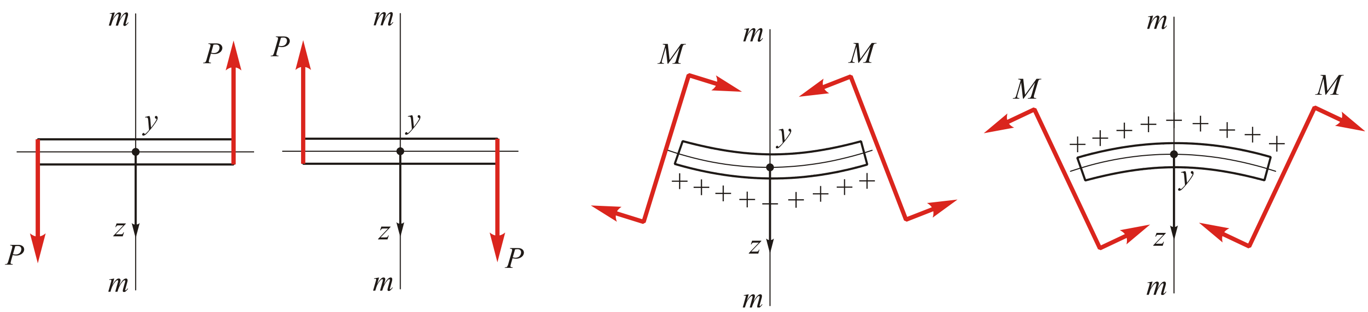
Shear force in a prismatic rod is equal to the algebraic sum of all external forces projections on the cross-section lying on one side of the section (left or right).

The bending moment at a section is equal to the sum of moments, in relevance to the transverse axis in the section, of all external forces applied to one side of the section (left or right).

Solution

1. Accepting the sign conventions in internal forces calculating.

(a) for shear force (b) for bending moment



$Q_z^{m-m} < 0$ $Q_z^{m-m} > 0$ $M_y^{m-m} > 0$ $M_y^{m-m} < 0$

Fig. 1

2. Calculating the reactions in supports R_A and R_C (see Fig. 2). Since their actual directions are unknown we will direct the reactions arbitrary, for example, upwards. The reaction positive sign from future calculating will mean that the reaction original direction is coincident with actual one and vice versa. For the reactions calculating we will use two momentum equations of equilibrium relative to supports (C and A points). Third equation of equilibrium in vertical direction we will use to check the result accuracy.

Note, that in designing the momentum balance equations clockwise rotation will be assumed to be positive and vice versa (see Fig. 2).

$$\sum M_A = 0 = +\frac{qa^2}{2} - M - R_C(a+b) - qa(\frac{a}{2}+b+c) + P(a+b+c),$$

$$R_C = \frac{1}{a+b}\left(-\frac{qa^2}{2} + M + qa(\frac{a}{2}+b+c) - P(a+b+c) \right) = +16,67\ \text{kN}.$$

$$\sum M_C = 0 = -\frac{qc^2}{2} - M + R_A(a+b) - qa(\frac{a}{2}+b) + Pc,$$

$$R_A = \frac{1}{a+b}\left(+\frac{qc^2}{2} + M + qa(\frac{a}{2}+b) - Pc \right) = +13.33\ \text{kN}.$$

$$\sum P_z = 0 = -R_A - R_C - qc + qa + P = -13.33 - 16.67 - 10\times 2 + 10\times 2 + 30 = 0.$$

3. Selecting the arbitrary cross-sections at *x*-distances from the origin of each potion and writing the equations of shear force and bending moment functions.

In this solution, we will consider the equilibrium of two left-situated parts of the rod (movement from left to right for portions I-I and II-II) and one right-situated part (movement from right to left for portion III-III). This is shown on Fig. 2. Note, that in such selection, the equations of internal forces will be the most simple in shape.

$$\text{I} - \text{I} \quad 0 < x < a:$$

$$Q_z^I(x) = R_A - qx\,|_{x=0} = 13.33\,|_{x=2} = 13.33 - 20 = -6.67\ \text{kN},$$

$$M_y^I(x) = R_A x - \frac{qx^2}{2}\,|_{x=0} = 0\,|_{x=2} = 26.66 - 20 = +6.66\ \text{kNm}.$$

Note, that the change of shear force sign within the boundaries of this section predicts the bending moment extreme value, since the derivative of bending moment is equal to shear force:

$$\frac{d(M_y^I(x))}{dx} = R_A - qx = \left| Q_z^I(x) \right|.$$

Therefore, zero shear force and also zero bending moment derivative represent the point of bending moment extreme value.

To find it, let us determine the coordinate of zero shear force x_e and substitute it into bending moment equation.

$$Q_z^I(x_e) = 0 = R_A - qx_e = 13.33 - 10x_e, \quad x_e = 1.33\ \text{m}.$$

$$M_{y_{\max}}^I = M_y^I(x_e) = R_A x_e - \frac{qx_e^2}{2} = 13.33\times 1.33 - \frac{10}{2}\times 1.33^2 = +8.89\ \text{kNm}.$$

$$\text{II} - \text{II} \quad 0 < x < b:$$

$$Q_z^{II}(x) = R_A - qa = 13.33 - 20 = -6,67\ \text{kN},$$

$$M_y^{II}(x) = R_A(a+x) - qa\left(\frac{a}{2} + x\right) - M\,|_{x=0} = 26.66 - 20 - 20 =$$

$$= 13.34\,|_{x=4} = 79.98 - 100 - 20 = -40\ \text{kNm}.$$

III – III $\quad 0 < x < c$:

$$Q_z^{III}(x) = P - qx\,|_{x=0} = 30\,|_{x=2} = 30 - 20 = 10 \text{ kN},$$

$$M_y^{III}(x) = -Px + \frac{qx^2}{2}\,|_{x=0} = 0\,|_{x=2} = -60 + 20 = -40 \text{ kNm}.$$

4. Designing the graphs of shear forces and bending moment distribution. Positive shear forces will be drawn upwards and vice versa. Bending moment graph will be drawn on tensile fibers according to the sign convention mentioned above (see Fig. 1). The graphs are shown on Fig. 2.

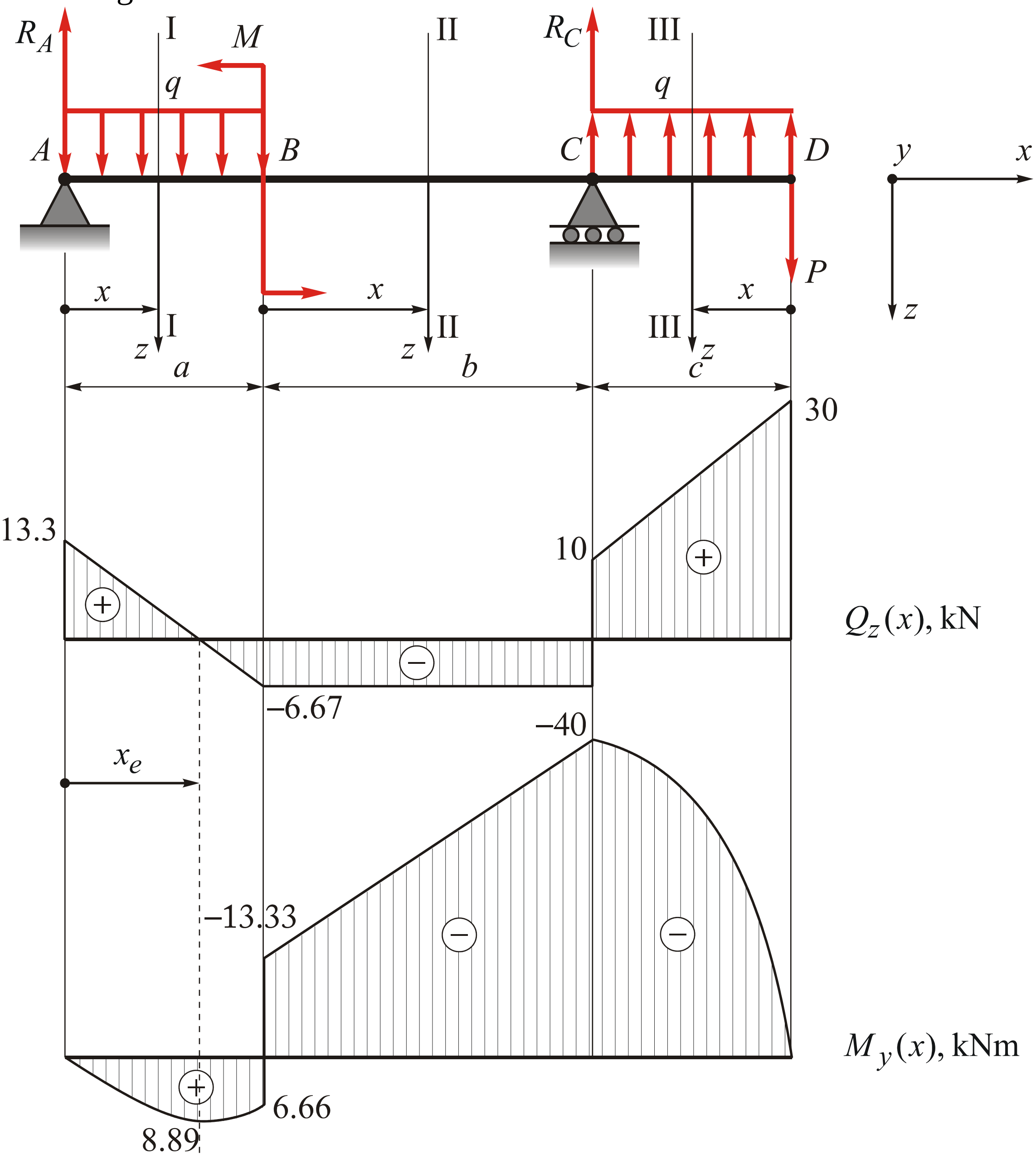


Fig. 2

5. Checking the results.

The "abrupts" on the Q_z graph are numerically equal to the force and reaction values in the points of these forces application.

The "abrupts" on the M_y graph are numerically equal to the concentrated moment values in the points of these moments application.

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Department of aircraft strength

Course
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HOME PROBLEM 6

Graphs of Shear and Normal Forces and Bending Moment Distribution in Plane Bending of Statically Determinate Frames

Name of student:

Group:

Advisor:

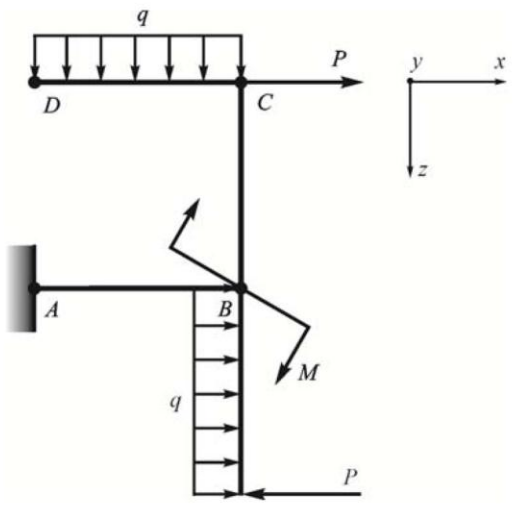
Data of submission:

Mark:

National aerospace university
"Kharkiv Aviation Institute"
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: graphs of shear force and bending moment distribution along the length of a beam in plane bending deformation.
Full name of the student, group

Variant: 1 **Complexity: 1**



Given: q = 10 kN/m, M = 30 kNm, P = 40 kN, l = 2 m.

Goal: obtain the equations of shear and normal forces and also bending moment in the cross-sections of a frame and design the graphs of their distribution along the frame portion length.

Full name of the lecturer **signature**

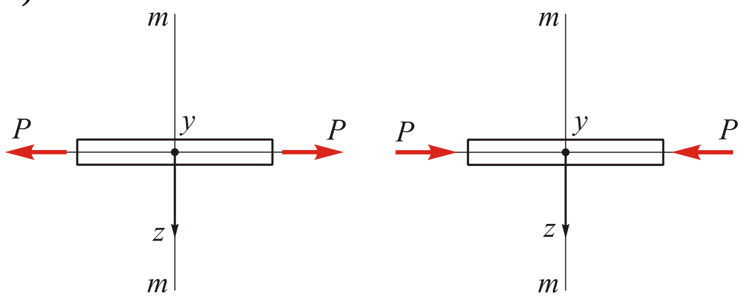
Mark:

In normal force calculating, we will use the rule that normal force in the cross-section is numerically equal to algebraic sum of external forces applied to the right or to the left part of the rod after its virtual cutting according to the method of sections. Tensile external force should be substituted into the equation with positive sign and visa versa. This sign convention is shown on Fig. 1.

Shear force in an arbitrary section is equal to the algebraic sum of all external forces projections on the z-axis of cross-section, but lying only on one side of the section (left or right) (see Fig. 1).

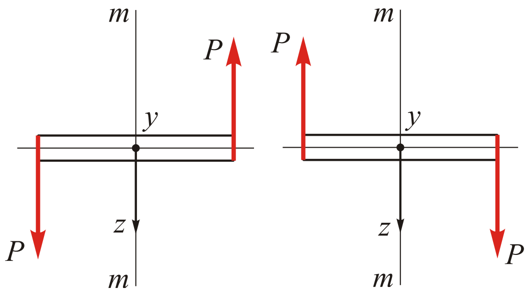
The bending moment in a section is equal to the sum of moments, in relevance to the transverse axis in the section, of all external forces applied to one side of the section (left or right) (see Fig. 1).

a) for normal force



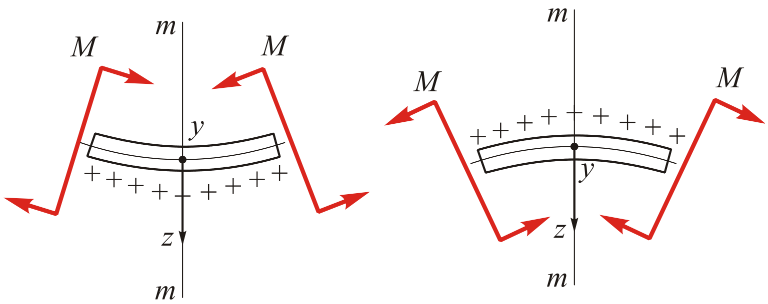
$N_x^{m-m} > 0$ $N_x^{m-m} < 0$

b) for shear force



$Q_z^{m-m} < 0$ $Q_z^{m-m} > 0$

б) for bending moments



$M_y^{m-m} > 0$ $M_y^{m-m} < 0$

Fig. 1

Comment: in the case when the curvature of deflected beam curve is coincident with z-axis direction, corresponding component of bending moment equation will be assumed to be positive and vice versa. The graph of bending moment will be designed on tensile fibers of the beam since position of tensile fiber is clear in both situations shown on Fig. 1.

Solution

1. Drawing the frame in scale and applying the support reactions in arbitrary directions.

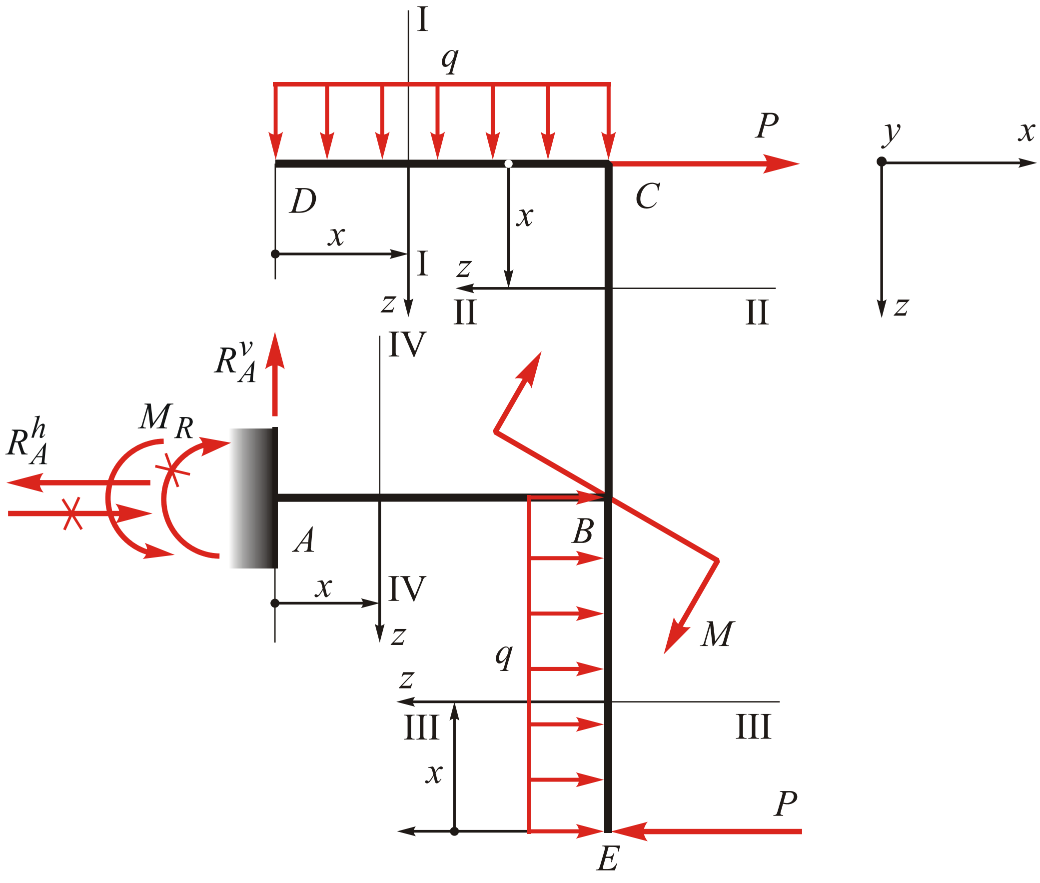


Fig. 2

2. Calculating the reactions in supports R_A^h, R_A^v, M_R.

Since the reactions actual directions are unknown we will direct the reactions arbitrary (see Fig. 2). The reaction positive sign from future calculating will mean that the reaction original direction is coincident with actual one and vice versa. In reactions calculating, we will use two momentum equations of equilibrium (relative to A and C points) and also equation of force equilibrium in vertical direction.

Note, that in designing the momentum equations of equilibrium clockwise rotation will be assumed to be negative and vice versa.

(1) $\sum M_A = -ql\frac{l}{2} - M - Pl - Pl + ql\frac{l}{2} - M_R = 0,$

$$M_R = -q\frac{l^2}{2} - M - 2Pl + \frac{ql^2}{2} = 30 - 80 - 80 = -190 \text{ kNm}.$$

"Minus" sign of M_R moment illustrates that its actual direction is opposite to preliminary assumed i.e. M_R is directed counterclockwise. It is shown on Fig. 2.

(2) $\sum M_D = -ql\frac{l}{2} + R_A^h l - M + ql\left(l + \frac{l}{2}\right) - 2Pl + M_R = 0,$

$$R_A^h = \frac{\frac{ql^2}{2} + M - ql\left(l + \frac{l}{2}\right) + 2Pl - M_R}{l} = \frac{20 + 30 - 60 + 160 - 190}{2} = -20 \text{ kN}.$$

"Minus" sign of R_A^h reaction illustrates that its actual direction is opposite to preliminary assumed i.e. R_A^h is directed to left. This is shown on Fig. 2.

(3) $\sum P_z = ql - R_A^v = 0, \quad R_A^v = ql = 10 \times 2 = +20$ kN.

3. Selecting the arbitrary cross-sections at *x*-distances from the origin of each potion and writing the equations of normal and shear forces and also bending moment functions.
In this solution, the portion balance will be considered to get the most simple equations of internal forces: the portions *I-I* and *II-II* will be considered from *D* point (motion from *D* to *B* point), potion *III-III* will be considered upward from *E* point and last portion will be considered from *A* point to right. This is shown on Fig. 2.

$I - I \quad (0 < x < l)$

$N_x^I(x) = 0$ kN,

$Q_z^I(x) = -qx\,|_{x=0} = 0\,|_{x=2} = -20$ kN,

$M_z^I(x) = -\dfrac{qx^2}{2}\,|_{x=0} = 0\,|_{x=2} = -20$ kNm.

$II - II \quad (0 < x < l)$

$N_x^{II}(x) = -ql = -20$ kN,

$Q_z^{II}(x) = +P = 40$ kN,

$M_y^{II}(x) = -\dfrac{ql^2}{2} + Px\,|_{x=0} = -20\,|_{x=2} = 60$ kNm.

$III - III \quad (0 < x < l)$

$N_x^{III}(x) = 0$ kN,

$Q_z^{III}(x) = +P - qx\,|_{x=0} = 40\,|_{x=2} = 20$ kN,

$M_y^{III}(x) = -Px + \dfrac{qx^2}{2}\,|_{x=0} = 0\,|_{x=2} = -80 + 20 = -60$ kNm.

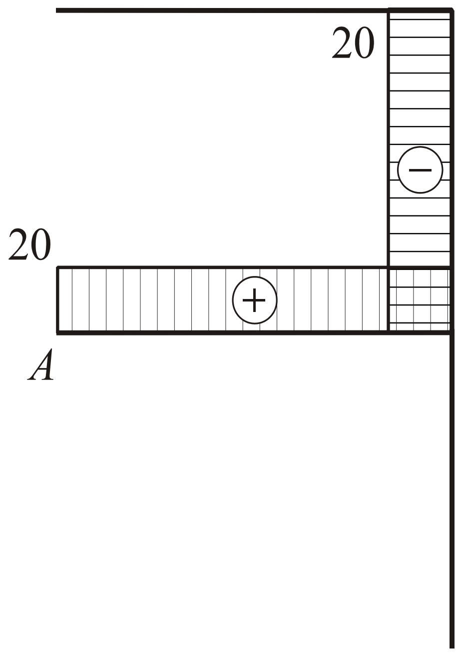
$IV - IV, \quad (0 < x < l)$

$N_x^{IV}(x) = +R_A^h = 20$ kN,

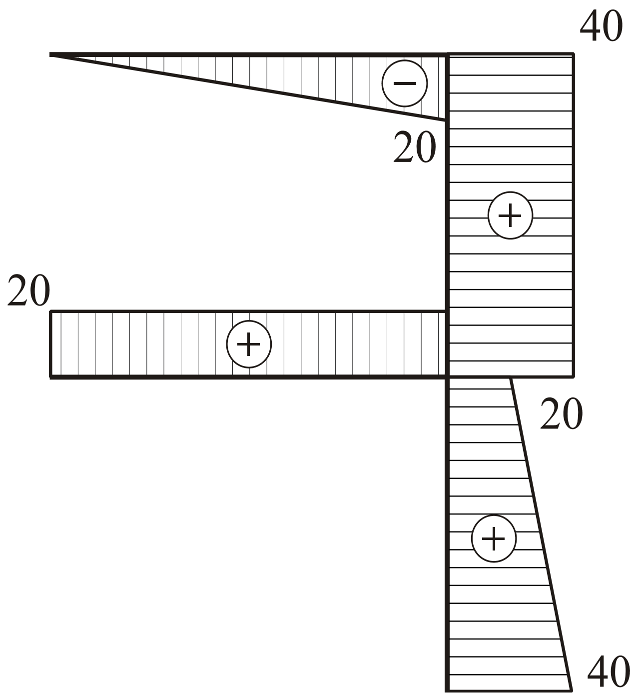
$Q_z^{IV}(x) = +R_A^v = 20$ kN,

$M_y^{IV}(x) = R_A^v x - M_R\,|_{x=0} = -190\,|_{x=2} = 40 - 190 = -150$ kNm.

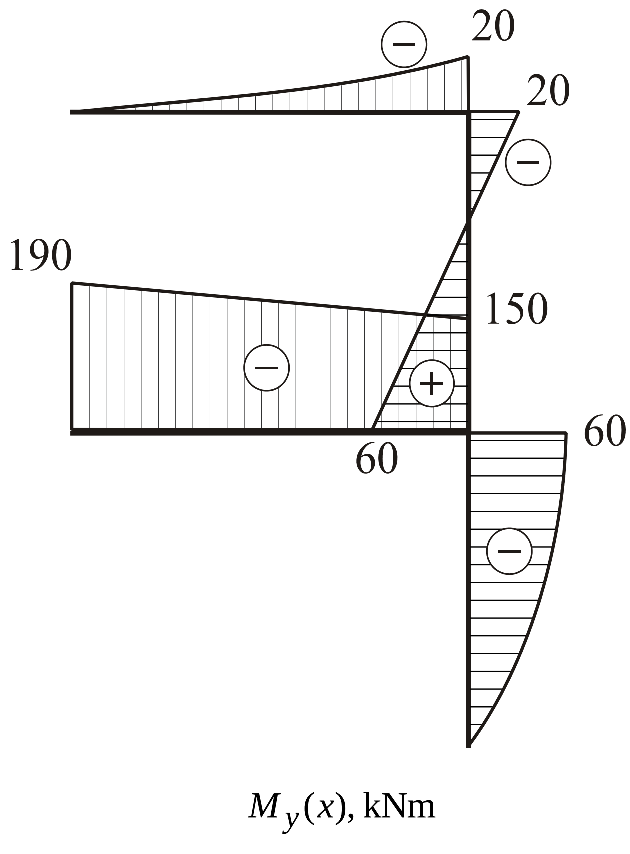
4. Designing the graphs of normal and shear forces and also bending moment distribution. Bending moment graph will be drawn on tensile fibers according to the sign convention mentioned above (see Fig. 1). The graphs are shown on Fig. 3.



$N_x(x)$, kN



$Q_z(x)$, kN



$M_y(x)$, kNm

Fig. 3

5. Checking the balance of two infinitely little elements of the frame.

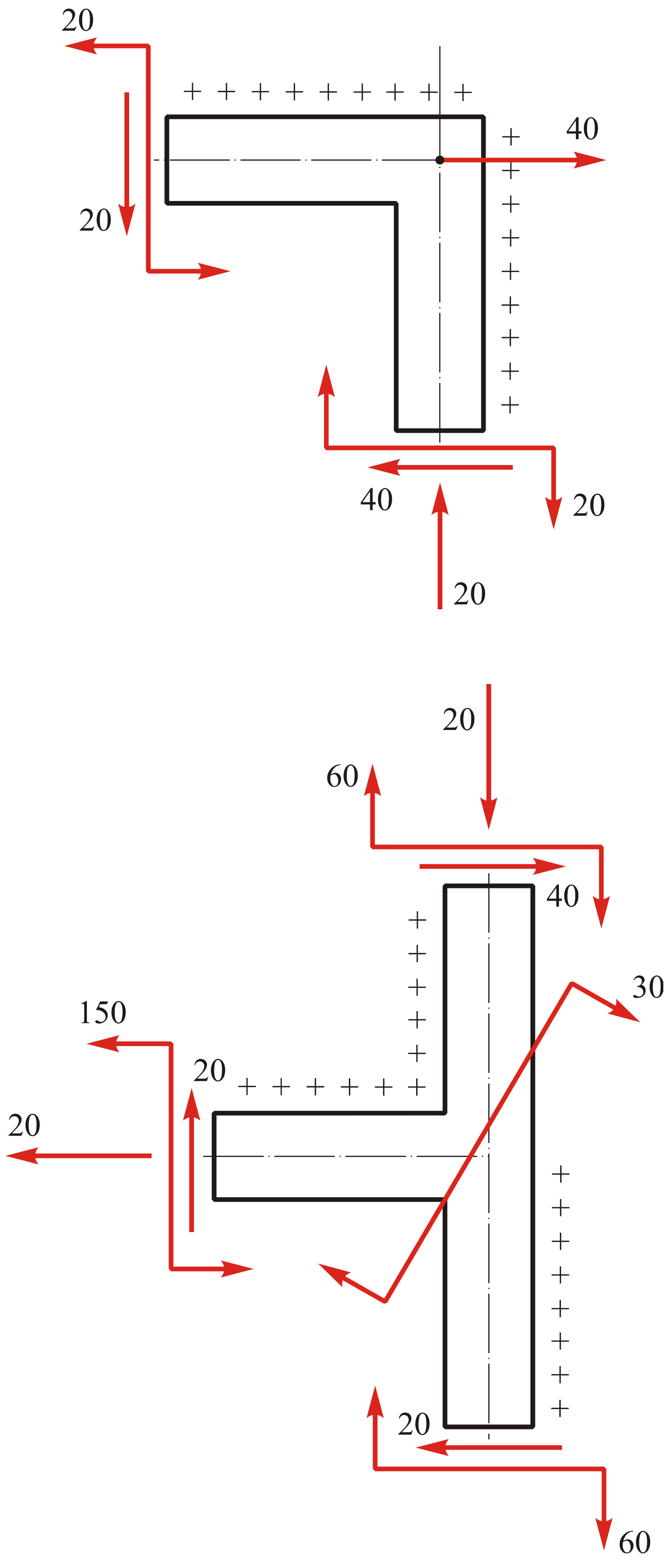


Fig. 4