LECTURE 7 Fundamentals of the Theory of Stress

1 State of Stress (Stress State) at a Point of Deformable Solid

A set of stresses occurring on all planes passing through the point in question is called the state of stress at the point.

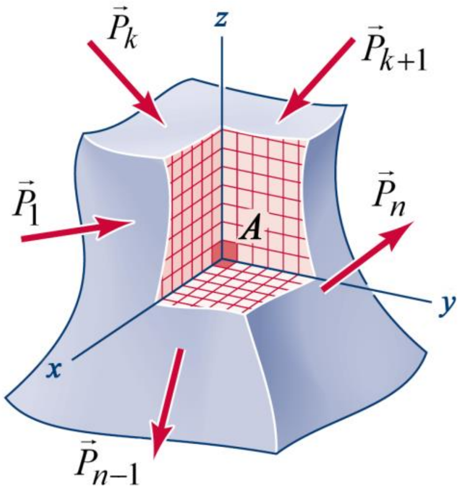

Fig.1

Suppose there is a certain body subjected to an arbitrary force system (Fig.1). We shall assume that the state of stress varies sufficiently slowly when passing from point to point of the body and it is always possible to choose in the vicinity of a point A sufficiently small area for which the **state of stress** may be considered homogeneous. It is apparent that such an approach is feasible only within the hypothesis of continuous medium. It allows passing to **infinitely small volume**.

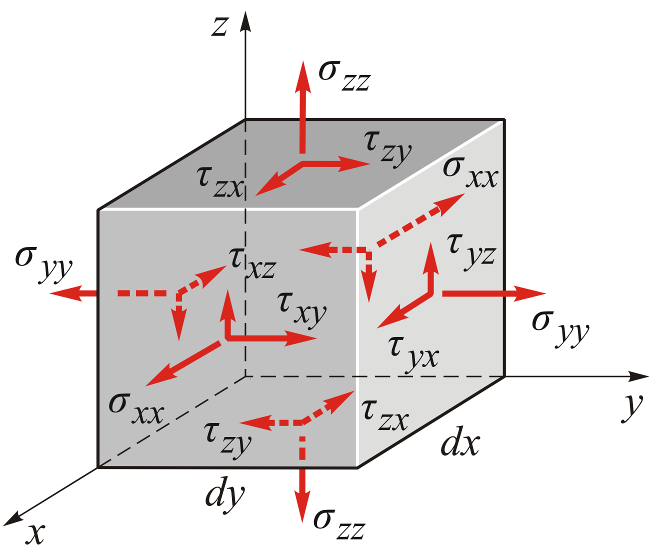

Fig. 2

To study the state of stress in particular point of the solid, we will isolate by six sections an **elementary volume** in the form of a right parallelepiped in the vicinity of the point in question.

If the dimensions of the parallelepiped are reduced, it will contract into this point. ***In the limit all faces of the parallelepiped pass through the point A***, and the stresses on the corresponding cutting planes may be regarded as the stresses at the point concerned.

The total stress acting on a cutting plane can be resolved into three components: one along the normal to the plane and two in the plane of the section (accordingly, normal and shear stresses).

Let us denote the normal stress by σ with a subscript corresponding to the appropriate axis (x, y or z). The shearing stress will be denoted by the symbol τ with ***two subscripts: the first corresponds to the axis perpendicular to the plane, and the second – to the axis along which the vector is directed***.

The stresses acting on three faces of the element (on three mutually perpendicular planes passing through the point) are shown in Fig. 2. The same stresses but opposite in sign occur on the hidden faces of the element.

2 Law of Equality for Shearing Stresses

The system of forces applied to the element must satisfy the conditions of equilibrium. Since the forces acting on the opposite faces are of different sign, the first three conditions of equilibrium are identically satisfied, and the sums of the projections of all forces on the x, y and z axes are zero. It remains to check whether the sums of the moments of all forces with respect to the x, y and z axes vanish.

Zero sum of the moments for the x axis is fulfilled if the moment of the force $\tau_{yz}dxdz$ is equal to the moment of the force $\tau_{zy}dxdy$, i.e.

$$\begin{aligned}
&\sum M_x(F)=0 \rightarrow \left(\tau_{zy}dxdy\right)dz-\left(\tau_{yz}dxdz\right)dy=0, \quad \text{then} \quad \tau_{zy}=\tau_{yz}, \\
&\sum M_y(F)=0 \rightarrow \left(\tau_{zx}dxdy\right)dz-\left(\tau_{xz}dydz\right)dx=0, \quad \text{then} \quad \tau_{zx}=\tau_{xz}, \qquad (1)\\
&\sum M_z(F)=0 \rightarrow \left(\tau_{xy}dydz\right)dx-\left(\tau_{yx}dxdz\right)dy=0, \quad \text{then} \quad \tau_{xy}=\tau_{yx}.
\end{aligned}$$

Thus, on two planes at right angles to each other the components of shearing stresses perpendicular to the **common edge** are equal and directed ***either both toward the edge or both away from the edge***. This is the **law of equality for shearing stresses**.

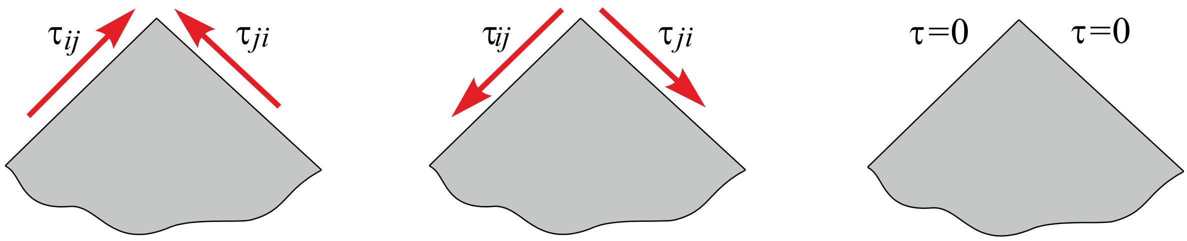

Fig. 3

3 Determination of Stresses on a Plane of General Position (Inclined Plane)

Given: σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx} (i.e. six stress components are given on three orthogonal planes). **It is required to determine the stresses on any plane passing through the given point**.

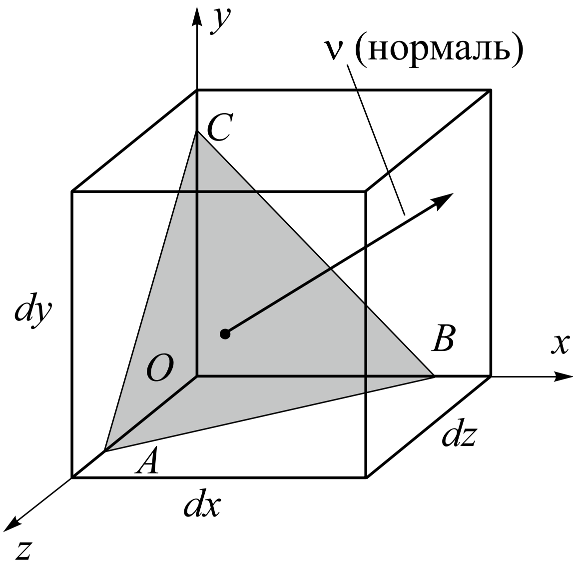

Fig. 4

We isolate an elementary volume *OABC* from a stressed body in the vicinity of point *O* (see Fig. 4). Three faces of the isolated element are coincident with the co-ordinate planes of the system *x*, *y*, *z* shown on Fig. 2. The fourth face is formed by a cutting **plane of general position** (*ABC* plane, inclined at three specified angles relative to orthogonal system of coordinates *x*, *y*, *z*). Its orientation in space will be defined by the **direction cosines** of the normal ν in the system of axes *x*, *y*, *z*, i.e. by quantities *l*, *m*, *n*.

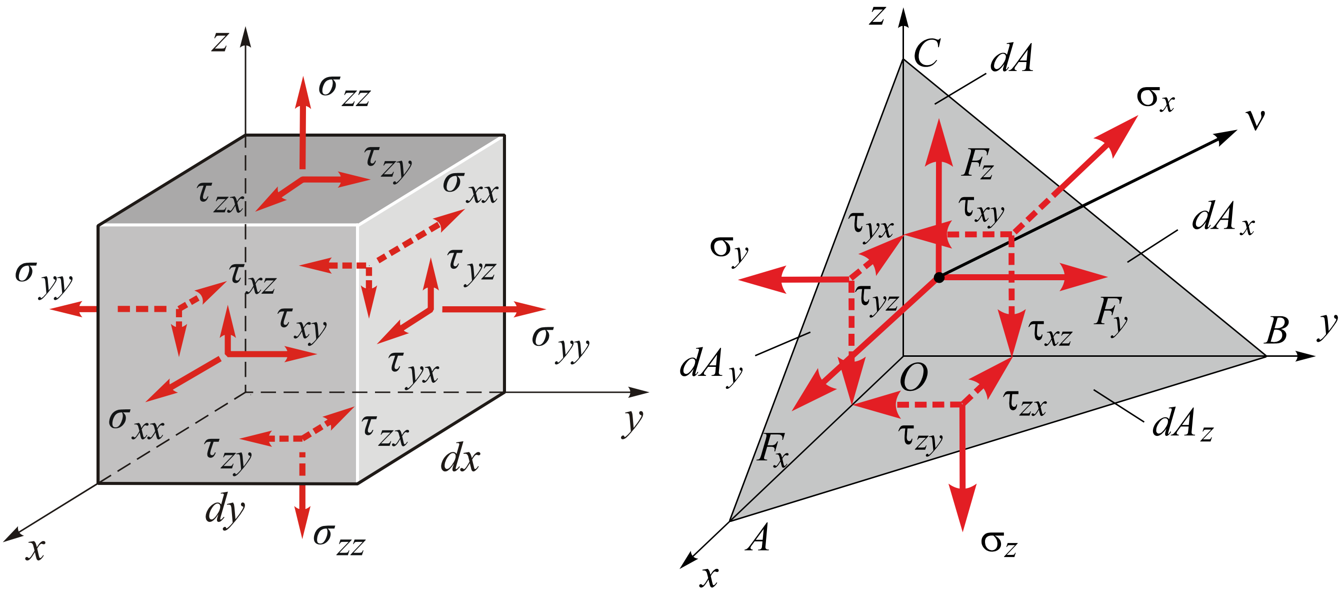

Fig. 5

We will project the total stress vector on the plane of general position *ABC* on the *x*, *y* and *z* axes. Let these projections be denoted by F_x, F_y, F_z, respectively.

Let the areas of the triangles *ABC*, *OBC*, *OAC*, *OAB* be denoted by dA, dA_x, dA_y, dA_z respectively. Obviously $dA_x = ldA$, $dA_y = mdA$, $dA_z = ndA$, where *l*, *m* and *n* are the direction cosines of the normal ν ($l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$).

By projecting all forces acting on the element on the *x*, *y* and *z* axes we obtain

$$\begin{aligned} F_x dA &= \sigma_x dA_x + \tau_{yx} dA_y + \tau_{zx} dA_z, \\ F_y dA &= \tau_{xy} dA_x + \sigma_y dA_y + \tau_{zy} dA_z, \\ F_z dA &= \tau_{xz} dA_x + \tau_{yz} dA_y + \sigma_z dA_z. \end{aligned} \tag{2}$$

or in accordance with relations mentioned above:

$$\begin{aligned} F_x &= \sigma_x l + \tau_{yx} m + \tau_{zx} n, \\ F_y &= \tau_{xy} l + \sigma_y m + \tau_{zy} n, \\ F_z &= \tau_{xz} l + \tau_{yz} m + \sigma_z n. \end{aligned} \tag{3}$$

The total, normal and shearing stresses on any plane passing through the point in question can easily be determined by the use of formulas

$$F_\nu = \sqrt{F_x^2 + F_y^2 + F_z^2}, \quad \sigma_\nu = F_x l + F_y m + F_z n, \quad \tau_\nu = \sqrt{F_\nu^2 - \sigma_\nu^2}. \tag{4}$$

Thus, indeed, the projections F_x, F_y and F_z for any plane defined by the direction cosines *l*, *m* and *n* are expressed in terms of six basic stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}. In another words, the ***state of stress at a point is defined by six components of stress***.

The state of stress represents so called **stress tensor**

$$T_\sigma = \begin{pmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}. \tag{5}$$

4 Principal Planes. Principal Stresses

At any chosen point in a stressed body there exists such a system of axes *x*, *y*, *z* in which the shearing stresses τ_{xy}, τ_{yz}, and τ_{zx} are zero. These axes are called **principal**

axes. The corresponding mutually perpendicular planes are called **principal planes**, and the normal stresses on them are called **principal stresses**. These stresses are denoted by σ_1, σ_2 and σ_2 in the order of increasing magnitude

$$\sigma_1 \geq \sigma_2 \geq \sigma_3. \quad (6)$$

It is algebraic relation. Principal stresses are shown in Fig. 6.

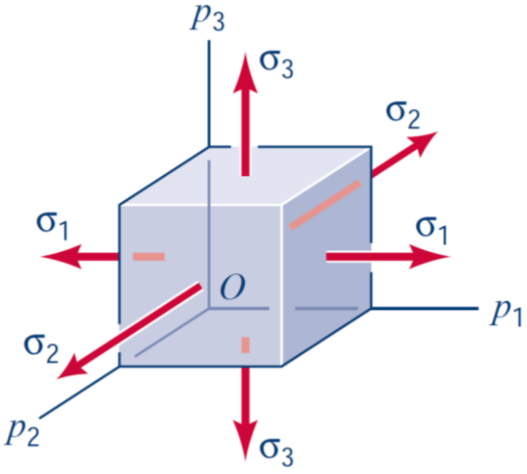

Fig. 6

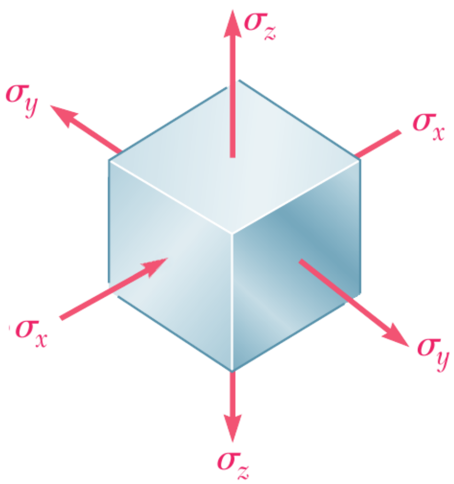

Fig. 7

Stress element under specified principal stresses is shown in Fig. 7. Let they are the following: $\sigma_x = -300\,\text{MPa}$, $\sigma_y = +150\,\text{MPa}$, $\sigma_z = +100\,\text{MPa}$. In such case, the stress subscript should be installed according (6):

$$\sigma_1 = +150\,\text{MPa},\ \sigma_2 = +100\,\text{MPa},\ \sigma_3 = -300\,\text{MPa}.$$

It was proved in elasticity theory, that three mutually perpendicular principal planes with principal stresses applied exist in each point of solid under an arbitrary loading.

5 Stress State Types

According to the principal stresses number it is usual to distinguish the following types of stress state (see Fig. 8): **uniaxial stress state** (Fig. 8, left), **biaxial (plane) stress state** (Fig. 8, middle), **triaxial stress state** (Fig. 8, right).

Note, that these stress states are illustrated under tension principal stresses, although different combinations between stress components directions takes place (see Figs. 9, 10, 11).

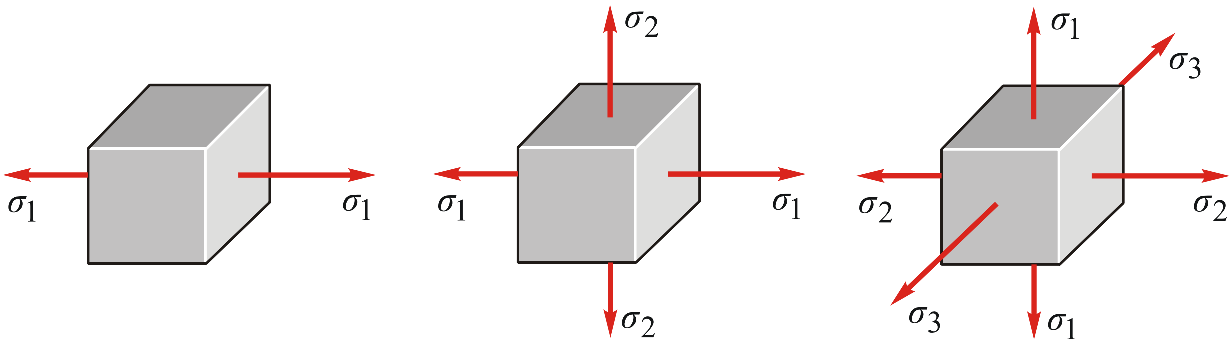

Fig. 8

1. **Uniaxial stress state**

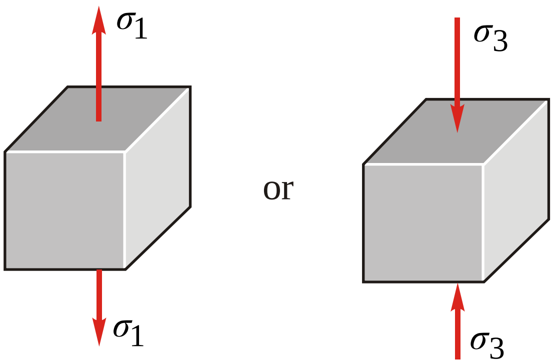

Fig. 9

$$\sigma_1 \neq 0,\ \sigma_2 = 0,\ \sigma_3 = 0; \qquad \sigma_1 = 0,\ \sigma_2 = 0,\ \sigma_3 \neq 0.$$

2. **Biaxial stress state**

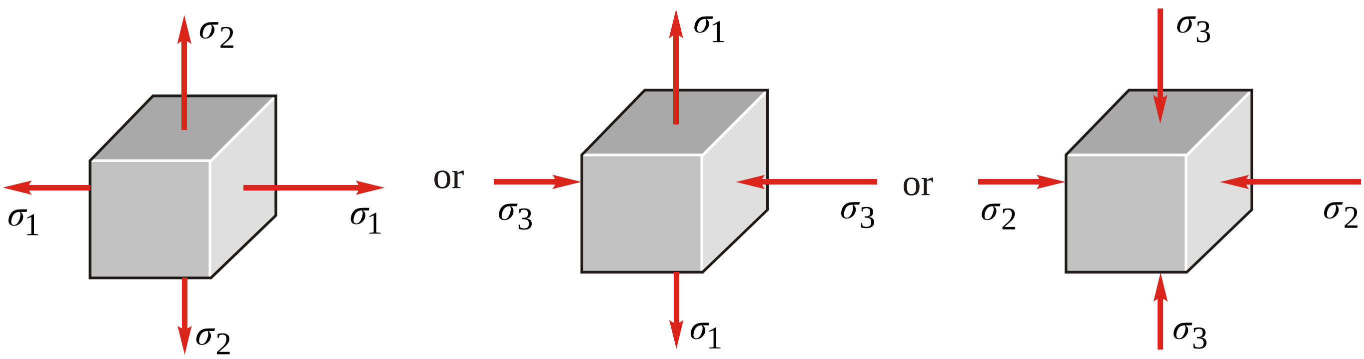

Fig. 10

3. **Triaxial stress state** (for example):

$\sigma_1 \neq 0$,

$\sigma_2 \neq 0$,

$\sigma_3 \neq 0$.

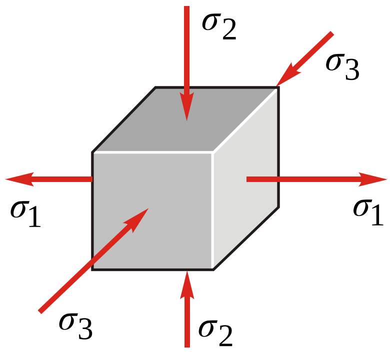

Fig. 11

The examples of uniaxial stress state are shown in Fig. 12 and the examples of biaxial stress state – in Fig. 13. Three dimensional stress state is illustrated by Fig. 14

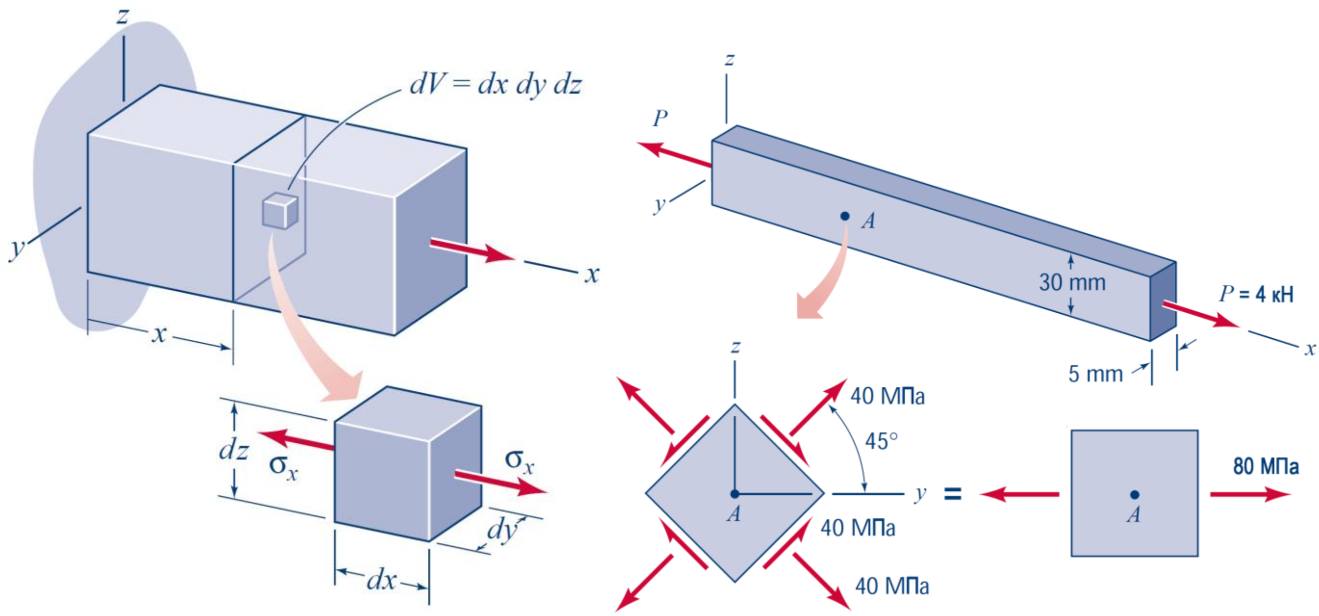

Fig. 12

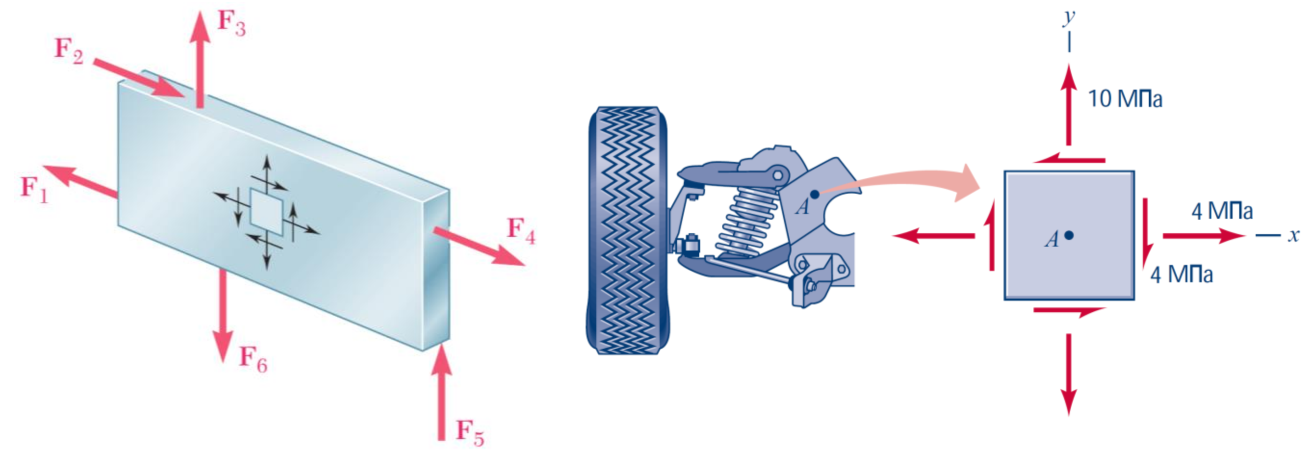
F1
F2
F3
F4
F5
F6
A
y
10 MПa
4 MПa
4 MПa
A
x

Fig. 13

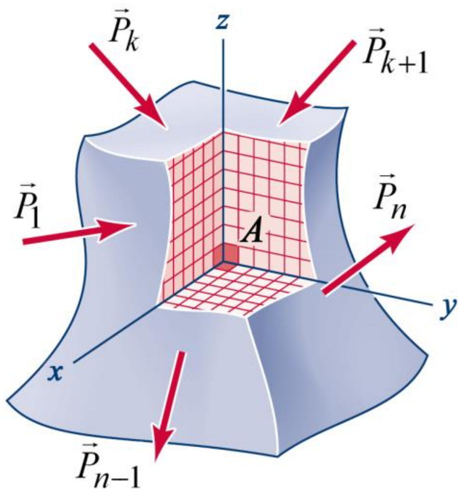
$\vec{P}_k$
z
$\vec{P}_{k+1}$
$\vec{P}_1$
A
$\vec{P}_n$
y
x
$\vec{P}_{n-1}$

Fig. 14