LECTURE 9 State of Strain at a Point of Deformable Solid

1 General Definition of Extensional and Shear Strain

Extensional strain is the change in length of a line segment divided by the original length of the line segment in the deformable solid under external loading (Fig. 1).

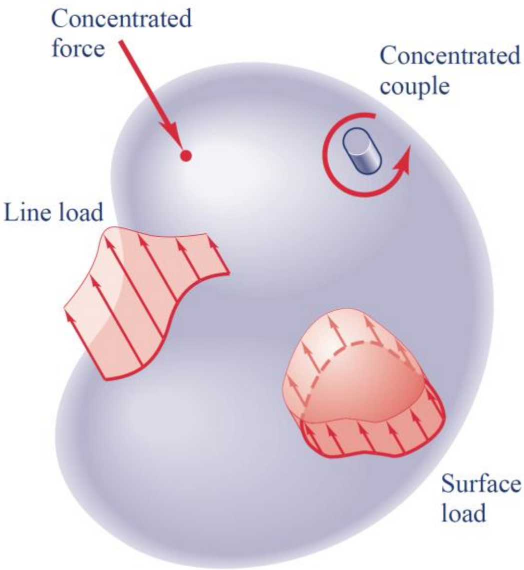
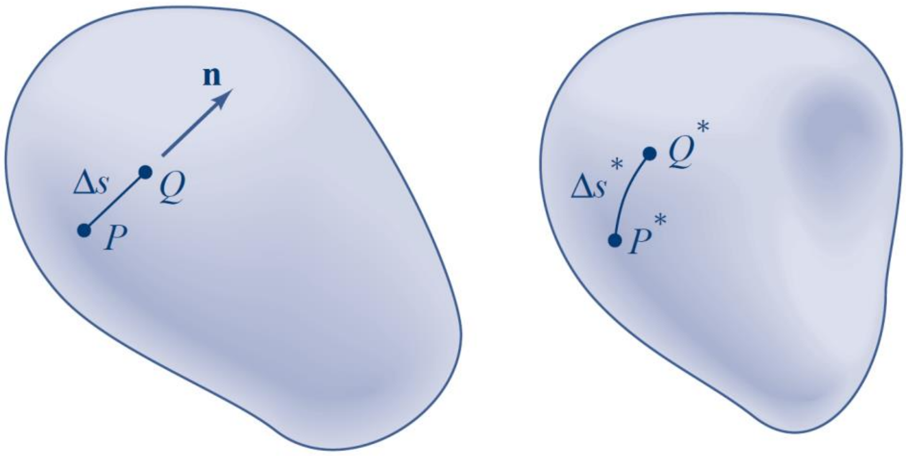


Fig. 1 Deformable solid under external loading

To define the extensional (linear) strain in a direction *n* at a point *P* in a body, we take an infinitesimal line segment of length Δs, in direction *n*, starting at *P* as shown in Fig. 2. That is, we take the infinitesimal line segment *PQ* of length Δs as the original line segment. After deformation, the line segment *PQ* becomes the infinitesimal arc *P*Q** with arclength Δs^*, as shown in Fig. 3. To determine the extensional strain right at point *P*, we need to start with a very short length Δs, that is, we must pick *Q* very close to point *P*. By picking *Q* closer and closer to *P*, we get, in the limit as $\Delta s \to 0$, the extensional strain right a point *P*. Then, the extensional strain at point *P* in direction *n*, denoted by $\varepsilon_n(P)$ is defined by

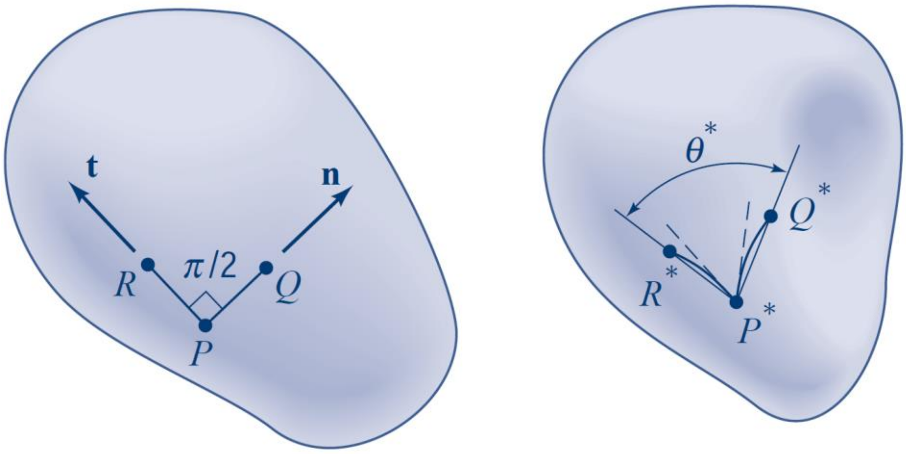
$$\varepsilon_n(P) = \lim_{\Delta s \to 0} \left(\frac{\Delta s^* - \Delta s}{\Delta s} \right). \quad (1)$$

When a body deforms, the change in angle that occurs between two line segments that were originally perpendicular to each other is called **shear strain**. To define the shear strain, let us consider the undeformed body in Fig. 3 (left) and the deformed body in Fig. 3 (right). Let *PQ* and *PR* be infinitesimal line segments in the *n* direction and *t* direction, respectively, in the undeformed body. After deformation, line segments PQ and PR become arcs P^*Q^* and P^*R^*. Secant lines P^*Q^* and P^*R^* define an angle θ^* in the deformed body.



(*a*) Undeformed body. (*b*) Deformed body.

Fig. 2 The infinitesimal line segment used to define extensional strain



(*a*) Undeformed body. (*b*) Deformed body.

Fig. 3 The angles used to define shear strain

In the limit, as we pick Q and R closer and closer to P, the angle θ^* approaches the angle between tangents to the arcs at P^*, shown as dashed lines in Fig. 3 (right). The shear strain between line segments extending from P in directions n and t is defined by the equation

$$\gamma(P) = \lim_{\substack{Q \to P \\ R \to P}} \left(90° - \theta^*\right) \tag{2}$$

It is important to note that extensional strain ε and shear strain γ vary with position in a body and with the orientation of the reference directions not only on the solid surface but also inside it. We will continue strain analysis applying the concept of strain element with infinitesimal dimensions.

2 Strain State at a Point of Deformable Solid

***A set of strains occurring along different axes and on different planes passing through a given point is called the state of strain at the point*.**

A change in shape and volume of a body is caused by **displacements** of its points. The displacements are caused by stresses applied to the faces of the stress element (see Fig. 4). The components of the total displacement vector along the *x*, *y* and *z* axes are denoted by *U*, *V* and *W* respectively.

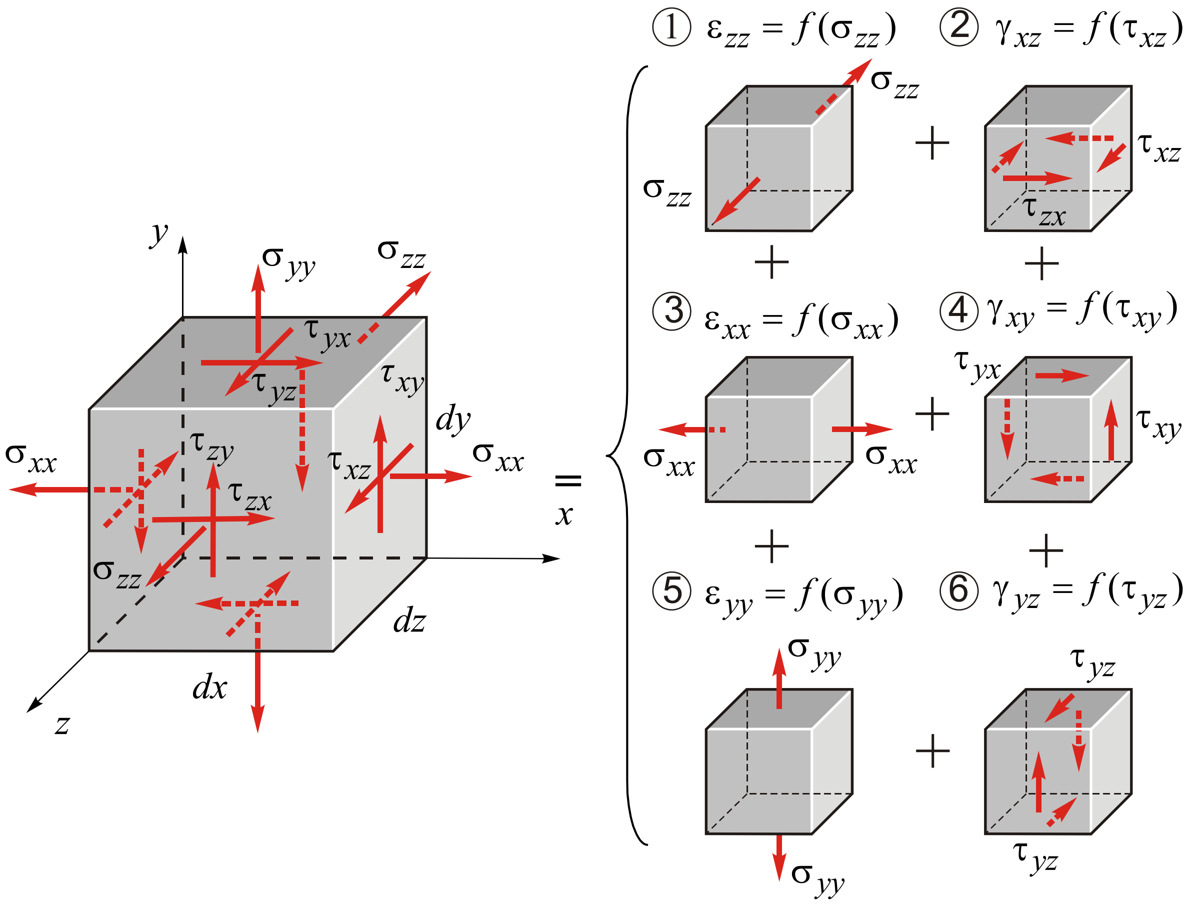


Fig. 4

Relationship between strain and displacement in uniaxial stress state.

In the most simple case of axial deformation illustrating on Fig. 5 first, third and fifth components of combined stress state, shown on Fig. 4 relationship between longitudinal strain and axial displacement is clear from Fig. 5. *A*-point is displaced along *x*-axis with *u* displacement, and *B*-point is displaced with increment *u*, i.e. with $u+du$ displacement. Elongation of *AB* segment is $A'B' - AB = (dx + u + du - u) - dx = du$. Then relative strain ε_x equals to

$$\varepsilon_x = \frac{du}{dx}. \qquad (3)$$

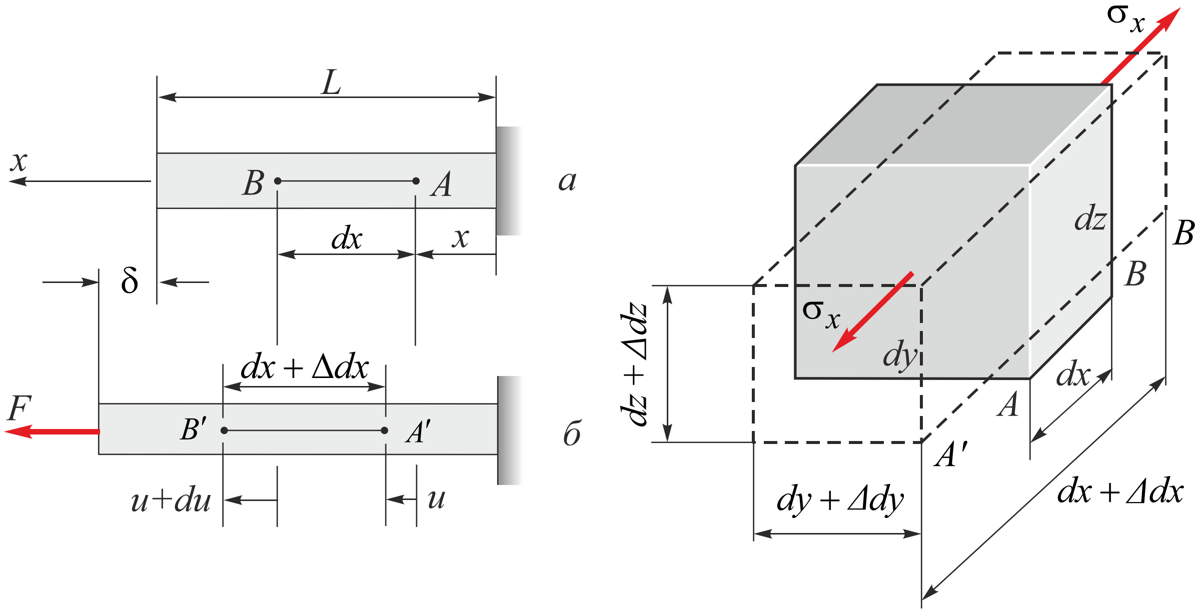


Fig. 5

Relationship between strain and displacement components in three-dimensional stress state.

Consider as the plane problem a segment *AB* whose direction coincides with the *x* axis (see Fig. 6). The distance between the points *A* and *B* is selected as infinitely little. Denote it by *dx*. The components of the displacement vector at the point *B* differ from those at the point *A* by amounts corresponding to the change in coordinate.

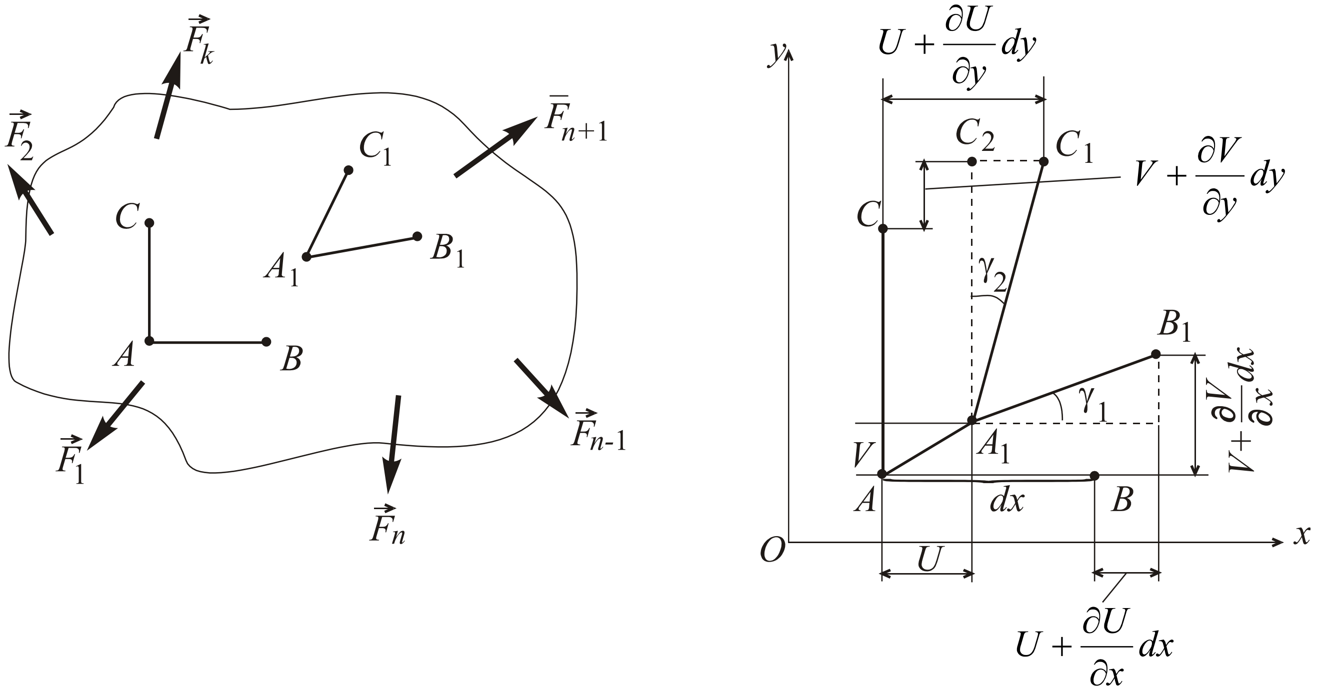


Fig. 6

The increment in length of the segment *AB* onto *x* axis is $\frac{\partial U}{\partial x}dx$. This increment is the **absolute elongation**, i.e.:

$$\Delta l_{AB} = A_1B_1 - AB = \frac{\partial U}{\partial x}dx\,. \tag{4}$$

The ratio of the length increment of the element *AB* to its original length is called longitudinal **relative deformation** or **linear strain** (relative elongation):

$$\varepsilon_x = \frac{\Delta l_{AB}}{AB} = \frac{\frac{\partial U}{\partial x}dx}{dx} = \frac{\partial U}{\partial x}\,. \tag{5}$$

Similarly,

$$\varepsilon_y = \frac{\partial V}{\partial y}; \qquad \varepsilon_z = \frac{\partial W}{\partial z}\,. \tag{6}$$

The angle of rotation of the segment *AB* in the *xy* plane is equal to the ratio of the difference between the displacement of the points *B* and *A* along the *x* axis to the length of the segment *dx*, i.e.,

$$\gamma_1 = \frac{\partial V}{\partial x}\,. \tag{7}$$

The angle of rotation of the segment *AC* in the *xy* plane is

$$\gamma_2 = \frac{\partial U}{\partial y}. \tag{8}$$

The sum of the angles γ_1 and γ_2 represents the change in right angle *BAC* i.e., the **angle of shear** in the *xy* plane

$$\gamma_{yx} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}. \tag{9}$$

The expressions for the angles of shear in the other two coordinate planes can be derived in a similar way.

Thus we have the following relations between displacements and strains at a point

$$\varepsilon_x = \frac{\partial U}{\partial x}; \quad \varepsilon_y = \frac{\partial V}{\partial y}; \quad \varepsilon_x = \frac{\partial W}{\partial z};$$

$$\gamma_{xy} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}; \quad \gamma_{yz} = \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}; \quad \gamma_{zx} = \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}. \tag{10}$$

The analysis of the state of strain shows that it possesses properties closely similar to those of the state of stress.

3 Hooke's Law

As has been established by the results of experimental tests (Figs 7–9) of specimens made from various structural materials, ***there exists a direct proportionality between normal stresses*** σ ***acting in cross-section and the strain*** ε, though certain limits (Fig. 8).

This relationship, which is the principal one in mechanics of materials is called **Hooke's law** and written as

$$\sigma = E\varepsilon, \tag{11}$$

where σ is normal stress, proportionality factor *E* is called the **modulus of elasticity (Young's modulus)**; ε is relative elongation (strain). It is necessary to know that the elasticity modulus is dependent on the temperature (see Fig. 10).

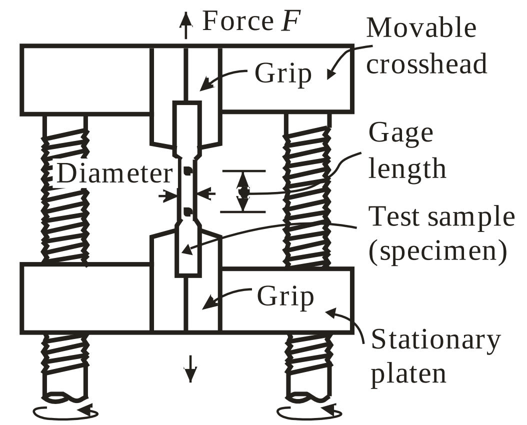


Fig. 7 Experimental device for material tensile mechanical tests

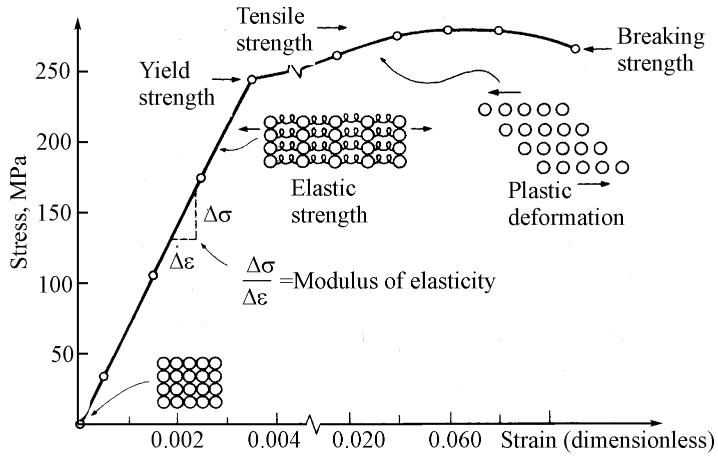


Fig. 8 Stress-strain curve (diagram) showing elastoplastic behavior of ductile material

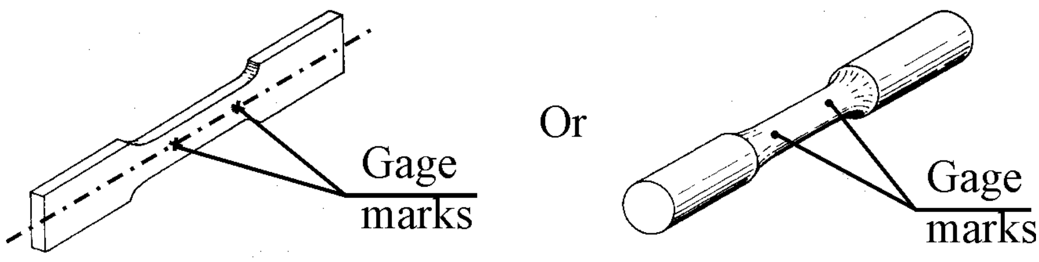


Fig. 9 Specimens used in tensile mechanical tests

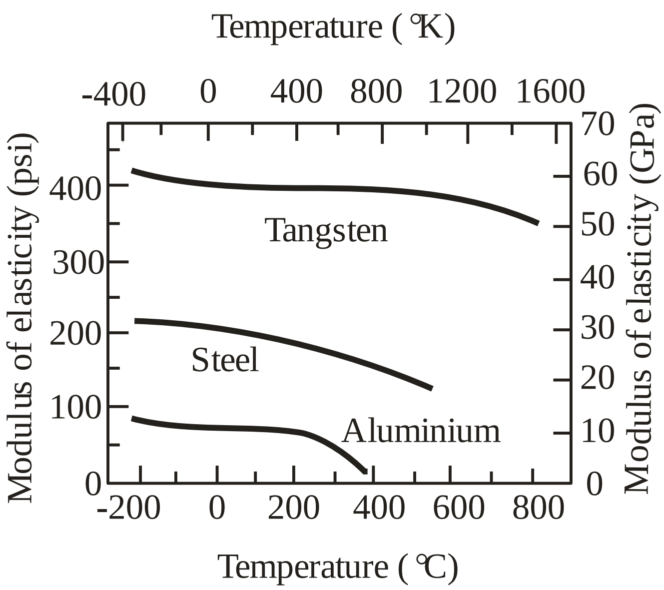


Fig. 10 Dependence of the Young's modulus on the temperature for different materials

Relative elongation is dimensionless and is often given as percentage of the original length:

$$\varepsilon\,\% = \frac{\Delta l}{l}100\%. \qquad (12)$$

The elastic modulus E is measured in the same units as the stress σ, i.e. in **Pascals** or **megaPascals** $\left(\mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2}\right)$. A stress of 1 Pa is very small. 1 MPa = 10^6 Pa (ten to the power of six).

The moduli of elasticity for various engineering materials are shown in Table 1.

Table 1 Modulus of Elasticity Values for Various Engineering Materials (Room-Temperature Conditions)

Material	Modulus of Elasticity	
	GPa	10^6 psi
1	2	3
METALS AND METAL ALLOYS **Plain Carbon and Low Alloy Steels**		
Steel alloy A36	207	30
Steel alloy 1020	207	30
Steel alloy 1040	207	30
Steel alloy 4140	207	30
Steel alloy 4340	207	30
Stainless alloy 304	193	28
Stainless alloy 316	193	28
Stainless alloy 405	200	29
Stainless alloy 440A	200	29
Stainless alloy 17-7PH	204	29.5
Cast Irons		
Gray irons		
• Grade G1800	66-97[a]	9.6-14[a]
• Grade G3000	90-113[a]	13.0-16.4[a]
• Grade G4000	110-138[a]	16-20[a]
Ductile irons		
• Grade 60-40-18	169	24.5
• Grade 80-55-06	168	24.4
• Grade 120-90-02	164	23.8
Aluminum Alloys		
Alloy 1100	69	10
Alloy 2024	72.4	10.5
Alloy 6061	69	10
Alloy 7075	71	10.3
Alloy 356.0	72.4	10.5
Copper Alloys		
C11000 (electrolytic tough pitch)	115	16.7
C17200 (beryllium-copper)	128	18.6
C26000 (cartridge brass)	110	16
C36000 (free-cutting brass)	97	14
C71500 (copper-nickel, 30%)	150	21.8
C93200 (bearing bronze)	100	14.5
Magnesium Alloys		
Alloy AZ31B	45	6.5
Alloy AZ91D	45	6.5
Titanium Alloys		
Commercially pure (ASTM grade 1)	103	14.9
Alloy Ti-5Al-2.5Sn	110	16

Alloy Ti-6Al-4V	114	16.5
Precious Metals		
Gold (commercially pure)	77	11.2
Platinum (commercially pure)	171	24.8
Silver (commercially pure)	74	10.7
Refractory Metals		
Molybdenum (commercially pure)	320	46.4
Tantalum (commercially pure)	185	27
Tungsten (commercially pure)	400	58
Miscellaneous Nonferrous Alloys		
Nickel 200	204	29.6
Inconel 625	207	30
Monel 400	180	26
Haynes alloy 25	236	34.2
Invar	141	20.5
Super invar	144	21
Kovar	207	30
Chemical lead	13.5	2
Tin (commercially pure)	44.3	6.4
Lead-Tin solder (60Sn-40Pb)	30	4.4
Zinc (commercially pure)	104.5	15.2
Zirconium, reactor grade 702	99.3	14.4
GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS		
Aluminum oxide		
• 99.9% pure	380	55
• 96%	303	44
• 90%	275	40
Concrete	25.4-36.6[a]	3.7-5.3[a]
Diamond		
• Natural	700-1200	102-174
• Synthetic	800-925	116-134
Gallium arsenide, single crystal	85	12.3
• In the (100) direction		
• In the (110) direction	122	17.7
• In the (111) direction	142	20.6
Glass, borosilicate (Pyrex)	70	10.1
Glass, soda-lime	69	10
Glass ceramic (Pyroceram)	120	17.4
Graphite		
• Extruded	11	1.6
• Isostatically molded	11.7	1.7
Silica, fused	73	10.6
Silicon, single crystal		

• In the (100) direction	129	18.7
• In the (110) direction	168	24.4
• In the (111) direction	187	27.1
Silicon carbide		
• Hot pressed	207-483	30-70
• Sintered	207-483	30-70
Silicon nitride		
• Hot pressed	304	44.1
• Reaction bonded	304	44.1
• Sintered	304	44.1
Zirconia, 3 mol% Y_2O_3	205	30
POLYMERS		
Elastomers	0.0034[b]	0.00049[b]
• Butadiene-acrylonitrile (nitrile)		
• Styrene-butadiene (SBR)	0.002-0.010[b]	0.0003-0.0015[b]
Epoxy	2.41	0.35
Nylon 6,6	1.59-3.79	0.230-0.550
Phenolic	2.76-4.83	0.40-0.70
Polybutylene terephthalate (PBT)	1.93-3.00	0.280-0.435
Polycarbonate (PC)	2.38	0.345
Polyester (thermoset)	2.06-4.41	0.30-0.64
Polyetheretherketone (PEEK)	1.10	0.16
Polyethylene		
• Low density (LDPE)	0.172-0.282	0.025-0.041
• High density (HDPE)	1.08	0.157
• Ultrahigh molecular weight (UHMWPE)	0.69	0.100
Polyethylene terephthalate (PET)	2.76-4.14	0.40-0.60
Polymethyl methacrylate (PMMA)	2.24-3.24	0.325-0.470
Polypropylene (PP)	1.14-1.55	0.165-0.225
Polystyrene (PS)	2.28-3.28	0.330-0.475
Polytetrafluoroethylene (PTFE)	0.40-0.55	0.058-0.080
Polyvinyl chloride (PVC)	2.41-4.14	0.35-0.60
FIBER MATERIALS		
Aramid (Kevlar 49)	131	19
Carbon (PAN precursor)		
• Standard modulus	230	33.4
• Intermediate modulus	285	41.3
• High modulus	400	58
E Glass	72.5	10.5
COMPOSITE MATERIALS		
Aramid fibers-epoxy matrix ($V_f = 0.60$)		
Longitudinal	76	11

Transverse	5.5	0.8
High modulus carbon fibers-epoxy matrix ($V_f = 0.60$)		
Longitudinal	220	32
Transverse	6.9	1.0
E glass fibers-epoxy matrix ($V_f = 0.60$)		
Longitudinal	45	6.5
Transverse	12	1.8
Wood		
• Douglas fir (12% moisture)		
Parallel to grain	10.8-13.6[c]	1.57-1.97[c]
Perpendicular to grain	0.54-0.68[c]	0.078-0.10[c]
• Red oak (12% moisture)		
Parallel to grain	11.0-14.1[c]	1.60-2.04[c]
Perpendicular to grain	0.55-0.71[c]	0.08-0.10[c]

[a] Secant modulus taken at 25% of ultimate strength.
[b] Modulus taken at 100% elongation.
[c] Measured in bending.

4 Lateral Strain. Poisson's Ratio

Experiments show that (within elasticity limits) the **extension** of a bar in the longitudinal direction is accompanied by its proportional **contraction** in the lateral direction (Fig. 11).

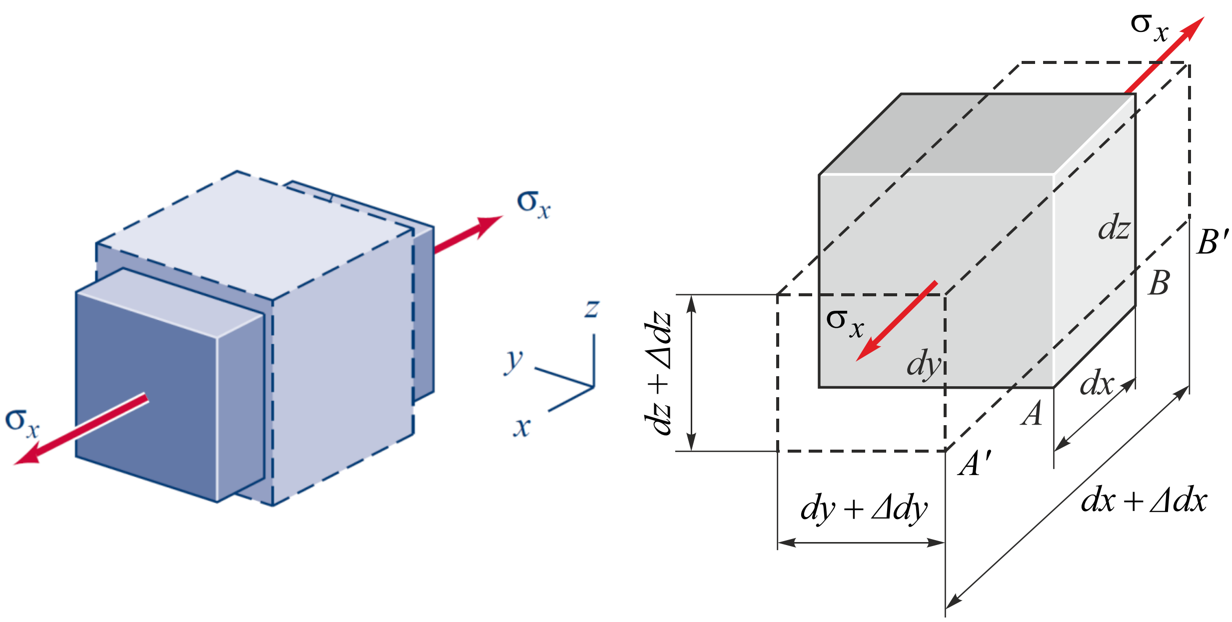


Fig. 11 Unidirectional tension of stress element and corresponding contraction of its edges in its lateral directions.

If we consider x-direction then

$$\varepsilon_{long} = \varepsilon_x = \frac{dx + \Delta dx - dx}{dx} = \frac{\Delta dx}{dx},$$
$$\varepsilon_{lat} = \varepsilon_y = \frac{dy + \Delta dy - dy}{dy} = \frac{\Delta dy}{dy}, \qquad (13)$$
$$\varepsilon_{lat} = \varepsilon_z = \frac{dz + \Delta dz - dz}{dz} = \frac{\Delta dz}{dz}.$$

As experiments show,

$$\varepsilon_{lat} = -\mu\varepsilon_{long} \quad \text{or} \qquad (14)$$

$$\varepsilon_y = -\mu\varepsilon_x,$$
$$\varepsilon_z = -\mu\varepsilon_x. \qquad (15)$$

where

$$\mu = \left|\frac{\varepsilon_{lat}}{\varepsilon_{long}}\right|. \qquad (16)$$

The absolute magnitude of this ratio is called the **Poisson's ratio**.

The quantity μ characterizes the properties of a material and is determined experimentally. The numerical values of μ lie within the limits from 0.25 … 0.35 for all metals and alloys. Poisson’s ratio values for various engineering materials are shown in Table 2.

Table 2 Poisson's Ratio Values for Various Engineering Materials (Room-Temperature Conditions)

Material	Poisson’s Ratio	Material	Poisson's Ratio
1	2	3	4
METALS AND METAL ALLOY Plain Carbon and Low Alloy Steels		**Refractory Metals**	
		Molybdenum (commercially pure)	0.32
Steel alloy A36	0.30		
Steel alloy 1020	0.30	Tantalum (commercially pure)	0.35
Steel alloy 1040	0.30		
Steel alloy 4140	0.30	Tungsten (commercially pure)	0.28

Steel alloy 4340	0.30		
Stainless Steels		**Miscellaneous Nonferrous Alloys**	
Stainless alloy 304	0.30	Nickel 200	0.31
Stainless alloy 316	0.30	Inconel 625	0.31
Stainless alloy 405	0.30	Monel 400	0.32
Stainless alloy 440A	0.30	Chemical lead	0.44
Stainless alloy 17-7PH	0.30	Tin (commercially pure)	0.33
Cast Irons		Zinc (commercially pure)	0.25
Gray irons		Zirconium, reactor grade 702	0.35
• Grade G1800	0.26	**GRAPHITE, CERAMICS, AND SEMICONDUCTING MATERIALS**	
• Grade G3000	0.26		
• Grade G4000	0.26	Aluminum oxide	
Ductile irons		• 99.9% pure	0.22
• Grade 60-40-18	0.29	• 96%	0.21
• Grade 80-55-06	0.31	• 90%	0.22
• Grade 120-90-02	0.28	Concrete	0.20
Aluminum Alloys		Diamond	
Alloy 1100	0.33	• Natural	0.10-0.30
Alloy 2024	0.33		
Alloy 6061	0.33	• Synthetic	0.20
Alloy 7075	0.33	Gallium arsenide	
Alloy 356.0	0.33	• (100) orientation	0.30
Copper Alloys		Glass, borosilicate (Pyrex)	0.20
C11000 (electrolytic tough pitch)	0.33	Glass, soda-lime	0.23
		Glass ceramic (Pyroceram)	0.25
C17200 (beryllium-copper)	0.30	Silica, fused	0.17

C26000 (cartridge brass)	0.35	Silicon	
C36000 (free-cutting brass)	0.34	• (100) orientation	0.28
C71500 (copper-nickel, 30%)	0.34	• (111) orientation	0.36
C93200 (bearing bronze)	0.34	Silicon carbide	
Magnesium Alloys		• Hot pressed	0.17
Alloy AZ31B	0.35	• Sintered	0.16
Alloy AZ91D	0.35	Silicon nitride	
Titanium Alloys		• Hot pressed	0.30
Commercially pure (ASTM grade 1)	0.34	• Reaction bonded	0.22
		• Sintered	0.28
Alloy Ti-5Al-2.5Sn	0.34	Zirconia, 3 mol% Y_2O_3	0.31
Alloy Ti-6Al-4V	0.34	**POLYMERS**	
Precious Metals		Nylon 6,6	0.39
Gold (commercially pure)	0.42	Polycarbonate (PC)	0.36
Platinum (commercially pure)	0.39	Polystyrene (PS)	0.33
		Polytetrafluoroethylene (PTFE)	0.46

Silver (commercially pure)	0.37	Polyvinyl chloride (PVC)	0.38
COMPOSITE MATERIALS			
Aramid fibers-epoxy matrix ($V_f = 0.6$)	0.34	**FIBER MATERIALS**	
		E Glass	0.22
High modulus carbon fibers-epoxy matrix ($V_f = 0.6$)	0.25	E glass fibers-epoxy matrix ($V_f = 0.6$)	0.19

5 Generalized Hooke's Law for three dimensional stress-strain state

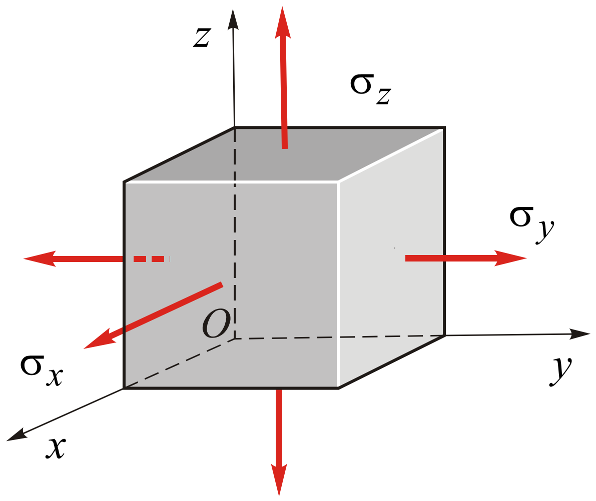


Fig. 12

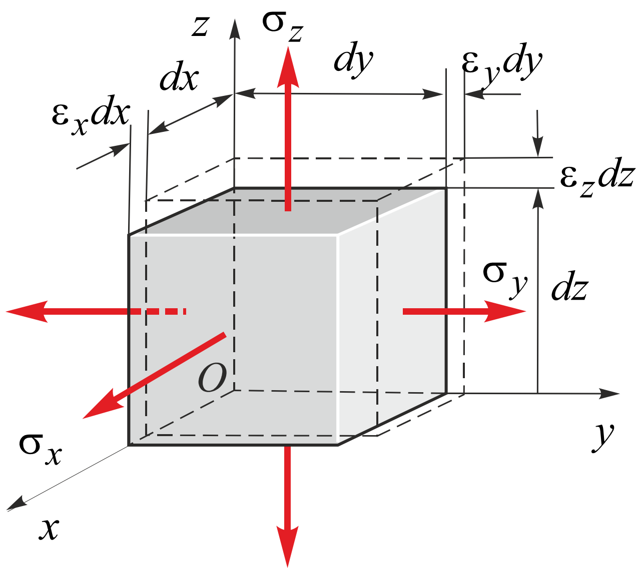


Fig. 13

Generalized Hooke's law represents the equations which connect components of stress and strain states. To write these equations, we will consider the stress element under the principal stresses loading (see Figs 12, 13). Our goal will be in finding the relative elongations (strains) of three mutually perpendicular edges along *x*, *y*, *z* axes. They will be denoted as ε_x, ε_y, ε_z (*x*, *y*, *z* directions are principal).

Given: σ_x, σ_y, σ_z, μ, E.

It is required to determine the quantities ε_x, ε_y, ε_z.

Let's note one more that the *x*, *y* and *z* axes coincide with the principal axes. The relative elongation (linear strain) in the direction of the *x* axis due to the stress σ_x is σ_x / E.

The stresses σ_y and σ_z produce elongations of opposite sign (lateral) along the x axis which are equal to $-\mu\frac{\sigma_y}{E}$ and $-\mu\frac{\sigma_z}{E}$. Consequently, total relative elongation (strain) with respect to the x axis

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}. \quad (17)$$

Similar expressions are obtained by analogy for ε_y and ε_z. Thus

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right], \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right], \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_y + \sigma_x\right)\right]. \end{aligned} \quad (18)$$

If the x, y and z axes don't coincide with the principal ones, then we will have

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right], \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right], \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_y + \sigma_x\right)\right]. \end{aligned} \quad (19)$$

but six additional relationships, connecting shear stresses and angular strains it is necessary to add in this case.

6 Relative Change of Volume (Unit Volume Change)

The initial volume of any infinitesimally small element before deformation (see Fig. 14):

$$dV_0 = dxdydz. \quad (20)$$

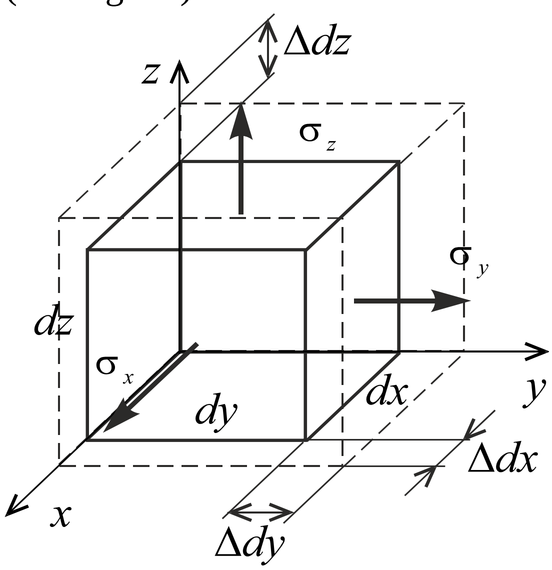


Fig. 14

After deformation, the lengths of the element increase to $dx + \Delta dx$, $dy + \Delta dy$ and $dz + \Delta dz$ (see Fig. 14). Thus, the volume of the element after deformation is

$$dV = \left(dx + \Delta dx\right)\left(dy + \Delta dy\right)\left(dz + \Delta dz\right). \quad (21)$$

The relative change of the volume (**unit volume change**)

$$e_V=\frac{dV-dV_0}{dV_0}=\frac{dV}{dV_0}-1=\left(\frac{dx+\Delta dx}{dx}\right)\left(\frac{dy+\Delta dy}{dy}\right)\left(\frac{dz+\Delta dz}{dz}\right)-1=$$

$$=(1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)-1=(1+\varepsilon_y+\varepsilon_x+\varepsilon_{xy})(1+\varepsilon_z)-1=$$

$$=1+\varepsilon_z+\varepsilon_y+\varepsilon_y\varepsilon_z+\varepsilon_x+\varepsilon_x\varepsilon_z+\varepsilon_x\varepsilon_y+\varepsilon_x\varepsilon_y\varepsilon_z-1\cong\varepsilon_x+\varepsilon_y+\varepsilon_z.$$

In result,

$$e_V=\varepsilon_x+\varepsilon_y+\varepsilon_z. \qquad (22)$$

Substituting expressions (17) into formula (20), we find

$$e_V=\frac{1}{E}\left[\sigma_x-\mu(\sigma_y+\sigma_z)\right]+\frac{1}{E}\left[\sigma_y-\mu(\sigma_x+\sigma_z)\right]+$$

$$+\frac{1}{E}\left[\sigma_z-\mu(\sigma_x+\sigma_y)\right]=\frac{1-2\mu}{E}\left[\sigma_x+\sigma_y+\sigma_z\right],$$

$$e_V=\frac{1-2\mu}{E}\left[\sigma_x+\sigma_y+\sigma_z\right], \qquad (23)$$

or

$$e_V=\frac{1-2\mu}{E}\left[\sigma_1+\sigma_2+\sigma_3\right] \qquad (24)$$

in terms of principal stresses.

7 Hook’s Law in Shear

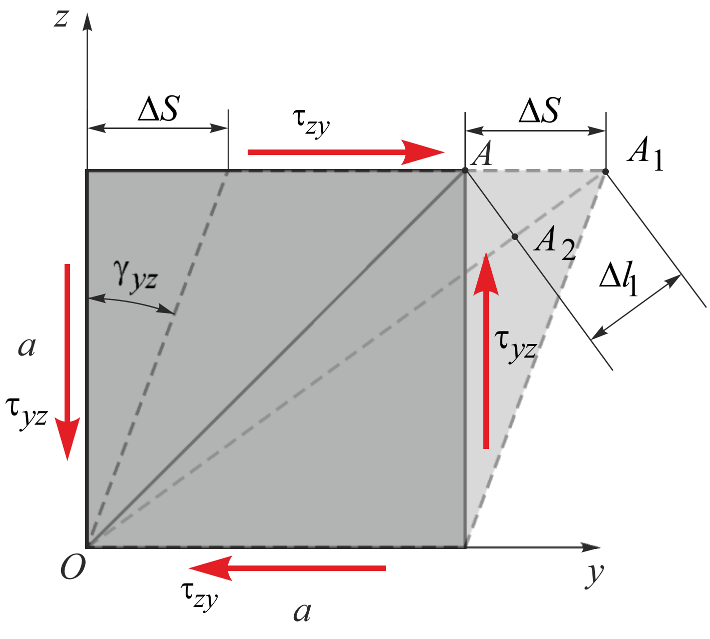


Fig. 15

Suppose now that there is a state of stress in which the faces of an isolated element are only acted on by shearing stresses τ (see Fig. 15). Such a state of stress is called **pure shear**. Principal stresses and directions in pure shear are situated at 45° relative to directions of shear (see Fig. 16). Principal stresses are equal $\sigma_1=+\tau_{yz}$ (in *OA* direction), $\sigma_2=0$, $\sigma_3=-\tau_{yz}$.

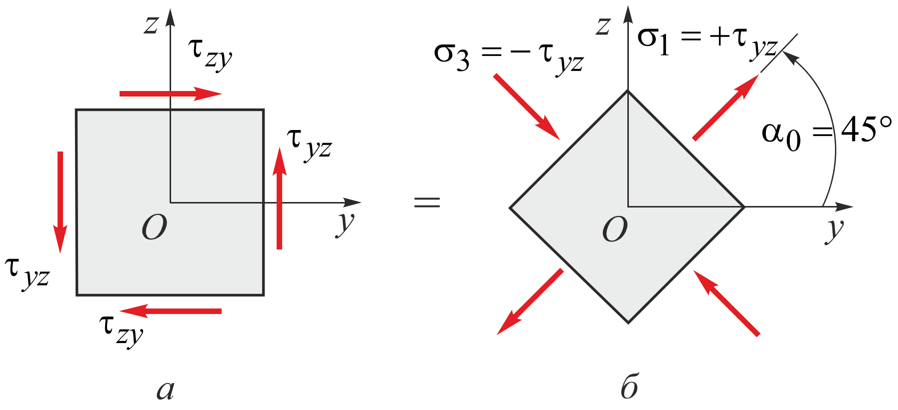


Fig. 16

It is seen from Fig. 15, that the quantity ΔS is **absolute shear**.

Let us calculate relative angle of shear:

$$\mathrm{tg}\gamma_{yz} \approx \gamma_{yz} = \frac{\Delta S}{a}. \tag{25}$$

This relationship is valid because $\Delta S \ll a$. Principal linear strain along OA direction is

$$\varepsilon_1 = \frac{\Delta l_1}{OA}. \tag{26}$$

Because

$$\Delta l_1 = \Delta S \cos 45° = \frac{\Delta S}{\sqrt{2}}, \tag{27}$$

and

$$OA = \sqrt{2}a, \tag{28}$$

we found that

$$\varepsilon_1 = \frac{\Delta S}{\sqrt{2}} \frac{1}{\sqrt{2}a} = \frac{1}{2}\frac{\Delta S}{a}. \tag{29}$$

Substituting (25) in (29) is resulted in

$$\varepsilon_1 = \frac{1}{2}\gamma_{xy}. \tag{30}$$

Otherwise, let us find the relative elongation ε_1 from generalized Hooke's law for pure shear deformation ($\sigma_1 = +\tau_{yz}$, $\sigma_2 = 0$, $\sigma_3 = -\tau_{yz}$):

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right] = \frac{1+\mu}{E}\tau_{yz}, \qquad (31)$$

Equating (30) and (31)

$$\frac{1+\mu}{E}\tau_{yz} = \frac{1}{2}\gamma_{yz}, \quad \tau_{yz} = \frac{E}{2(1+\mu)}\gamma_{yz},$$

where proportionality factor

$$G = \frac{E}{2(1+\mu)} \qquad (32)$$

is called **modulus in shear (shear modulus)**

Finally, we have

$$\tau_{yz} = G\gamma_{yz}. \qquad (33)$$

This relationship is called **Hook's law in shear**.

8 Strain Energy and Strain Energy Density

Strain energy is a fundamental concept in applied mechanics, and strain-energy principles are widely used for determining the response of machines and structures to loads. Now we introduce the subject of strain energy in its simplest form by considering only axially loaded members subjected to static loads.

To illustrate the basic ideas, let us consider a prismatic bar subjected to a tensile force (see Fig. 17). We assume that the load is applied slowly, so that is gradually increases from 0 to its maximum value P. Such a load is called a **static load** because there are no dynamic or inertial effects due to motion. The bar gradually elongates as the load is applied, eventually reaching its maximum elongation δ at the same time that the load reaches its full value P. There after, the load and elongation remain unchanged.

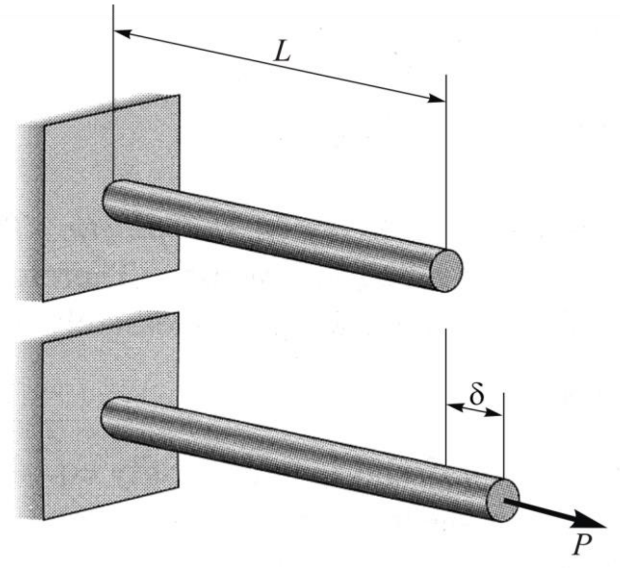


Fig. 17 Prismatic bar subjected to a statically applied load

During the loading, the load P moves slowly through the distance δ and does a certain amount of **work**. To evaluate this work, we recall from elementary mechanics that a constant force does work equal to the product of the force at the distance through which it moves. However, in our case ***the force varies in magnitude from 0 to its maximum value P***. To find the work done by the load under these conditions, we need to know the manner in which the force varies. This information is supplied by a **load-displacement diagram**, which is plotted for linearly elastic material in Fig. 18. On this diagram the vertical axis represents the axial load and the horizontal axis represents the corresponding elongation of the bar.

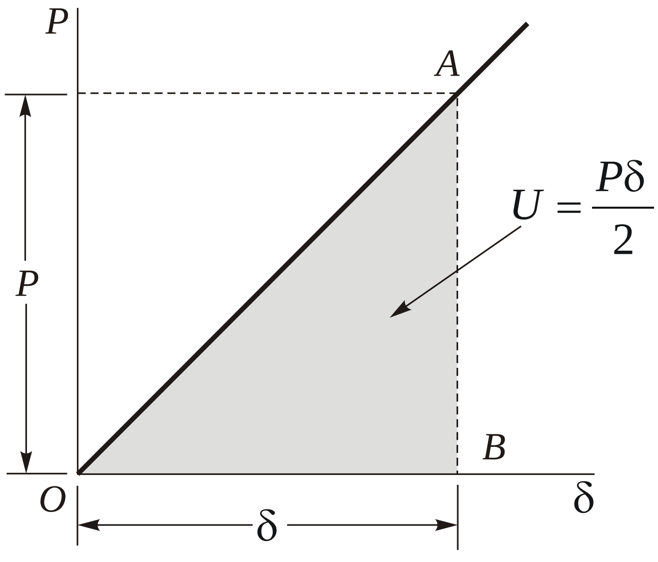


Fig. 18 Load-displacement diagram for a bar of linearly elastic material.

The work done by the load is represented in the figure by the area of the shaded strip below the load-displacement curve. In geometric terms, ***the work done by the load is equal to the area below the load-displacement curve***.

When the load stretches the bar, strains are produced. The presence of these strains increases the energy level of the bar itself. Therefore, a new quantity, called **strain energy**, is defined as the energy absorbed by the bar during the loading process. From the principle of conservation of energy, we know that this strain energy is equal to the work done by the load provided no energy is added or subtracted in the form of heat. Therefore,

$$U = W = \frac{P\delta}{2} \tag{34}$$

in which U is the symbol of strain energy, W is symbol of work. Sometimes strain energy is referred to as **internal work** to distinguish it from the external work done by the load.

In Fig. 18, strain energy is the area of the shaded triangle *OAB*. The principle that the work of the external loads is equal to the strain energy for the case of linearly elastic behavior was first stated by the French engineer B.P.E. Clapeyron and is known as **Clapeyron theorem**.

Since Hooke's law establishes linear relationship between the load and elongation or between the stress and strain in linearly elastic material, elongation of the bar $\Delta l = l - l_0 = \delta$ is given by the equation

$$\delta = \frac{PL}{EA}, \tag{35}$$

Because $\sigma = P/A$, $\varepsilon = \Delta l / l_0$, $\sigma = E\varepsilon$.

Combining this equation with Eq. 34 enables us to express the strain energy for a linearly elastic bar in the following form

$$U = \frac{P^2 L}{2EA}. \tag{36}$$

In many situations it is convenient to use a quantity called **strain-energy density**, defined as the strain energy per unit volume of material. Expressions for strain-energy density in the case of linearly elastic materials can be obtained from the formula for strain energy of a prismatic bar (Eq. 36). Since the strain energy of the bar is distributed uniformly throughout its volume, we can determine the strain-energy density by dividing the total strain energy U by the volume AL of the bar. Thus, the strain-energy density, denoted by the symbol U_0, can be expressed in the form

$$U_0 = \frac{P^2}{2EA^2}. \tag{37}$$

If we replace P/A by the stress σ, we get

$$U_0 = \frac{\sigma^2}{2E} = \frac{1}{2}\sigma\varepsilon. \tag{38}$$

This equation give the strain-energy density in a linearly elastic material in terms of the normal stress σ and the normal strain ε. This equation corresponds to uniaxial stress state at a point of the bar.

The strain-energy stored in an elementary volume of elastic solid in its deformation in three-dimensional stress-state is determined by the total work done by the internal forces distributed over the surface of this stress element. The normal force $\sigma_x dydz$ does work on the displacement $\Delta dx = \varepsilon_x dx$ (see Fig. 13). This work of internal elastic force is given by formula

$$\frac{1}{2}\Delta dx(\sigma_x dydz)\left\{\frac{\Delta dx}{dx} = \varepsilon_x\right\} = \frac{1}{2}\sigma_x\varepsilon_x dxdydz, \tag{39}$$

where ε_x is the strain along the *x* axis.

Similar expressions are obtained for the work done by two other normal components: $\frac{1}{2}\sigma_y dzdx\varepsilon_y dy$; $\frac{1}{2}\sigma_z dxdy\varepsilon_z dz$.

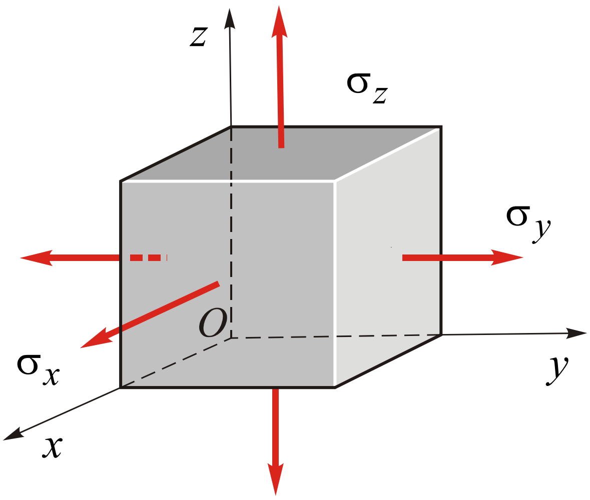


Fig. 19

Then total strain energy stored in an elementary volume $dxdydz$ (see Fig. 19) is:

$$dU = \frac{1}{2}dxdydz\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z\right). \tag{40}$$

For this stress element, strain-energy density

$$U_0 = \frac{dU}{dV} = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z\right). \tag{41}$$

Let's express strains in terms of stresses by generalized Hook's law:

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right],$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right], \qquad (42)$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_x + \sigma_y\right)\right].$$

Then

$$U_0 = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\mu\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right)\right]. \qquad (43)$$

Conclusion. ***Strain energy describes the possibility of elastic solid to change its shape and also volume in deformation.***

Example 1. Elastic aluminum block *R* is confined between plane parallel walls of absolutely rigid block *S* (see Fig. 20). A uniformly distributed pressure *p* is applied to the top of the block by a resultant force *F*. Disregarding friction between the blocks find stresses on the *R* block faces and strains of its edges. Calculate also its relative change in volume (dilatation).

Given: block *R* dimensions: $(1\times1\times1)\times10^{-2}$ m. Its mechanical properties: $E = 70$ GPa, $\mu = 0.3$. Pressure $p = 60$ MPa.

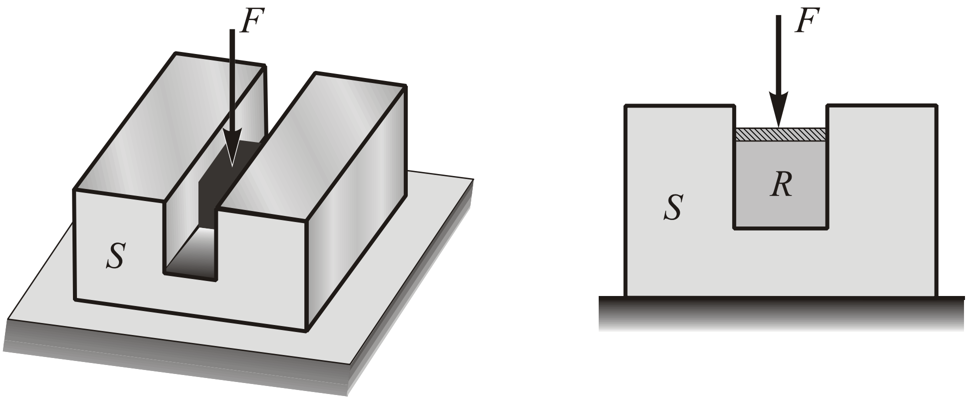


Fig. 20

Due to disregarding friction, elastic block will be under principal stresses. Its stress state may be described by Fig. 21. It is evident that $\sigma_z = -60\,\text{MPa}$, $\varepsilon_y = 0$, $\sigma_x = 0$, $\varepsilon_z < 0$, $\sigma_y < 0$, $\varepsilon_x > 0$.

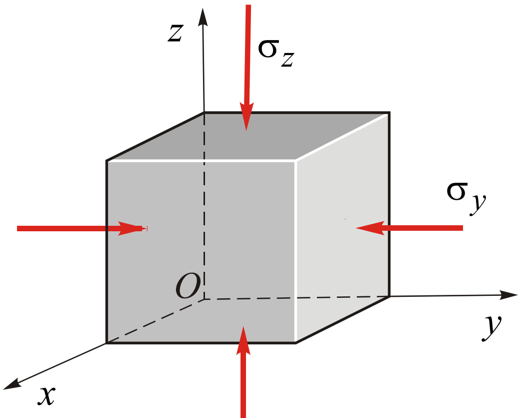


Fig. 21

Three unknown components of stress–strain state we will determine using the equations of generalized Hooke's law (19):

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right],$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right],$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_y + \sigma_x\right)\right],$$

or for our case (in MPa):

$$\begin{cases} \varepsilon_x = \dfrac{1}{70\times10^3}\left[0 - 0.3\times\left(\sigma_y + (-60)\right)\right], \\ \varepsilon_y = 0 = \dfrac{1}{70\times10^3}\left[\sigma_y - 0.3\times\left(0 + (-60)\right)\right], \\ \varepsilon_z = \dfrac{1}{70\times10^3}\left[(-60) - 0.3\times\left(0 + \sigma_y\right)\right]. \end{cases}$$

In result of solution we get

$$\sigma_y = -18\,\text{MPa}, \quad \varepsilon_x = +33.4\times10^{-3}, \quad \varepsilon_z = -78\times10^{-3}.$$

Relative change in volume is

$$e_V = (33.4 + 0 - 78)\times10^{-3} = -44.6\times10^{-3} \text{ (volume decreases).}$$

Strain energy density is determined by formula (43):

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right].$$

In our case of loading (in MPa):

$$U_0 = \frac{1}{2\times70\times10^3}\left[0 + (-18)^2 + (-60)^2 - 0.6(0 + (-18)\times(-60) + 0)\right] = 65.3\times10^{-3}\,\text{G/m}^3.$$

Example 2 An elastic aluminum block R with dimensions $(1\times1\times1)\times10^{-2}$m (Fig. 22) is compressed inside of an absolutely rigid array by the force F that applies a uniformly distributed pressure to the block equals to 60 MPa. Its elastic properties are $E = 70\,\text{GPa}$, $\mu = 0.3$. Determine stresses on the R block faces and strains of its edges. Calculate also its relative change in volume (dilatation) and strain energy density.

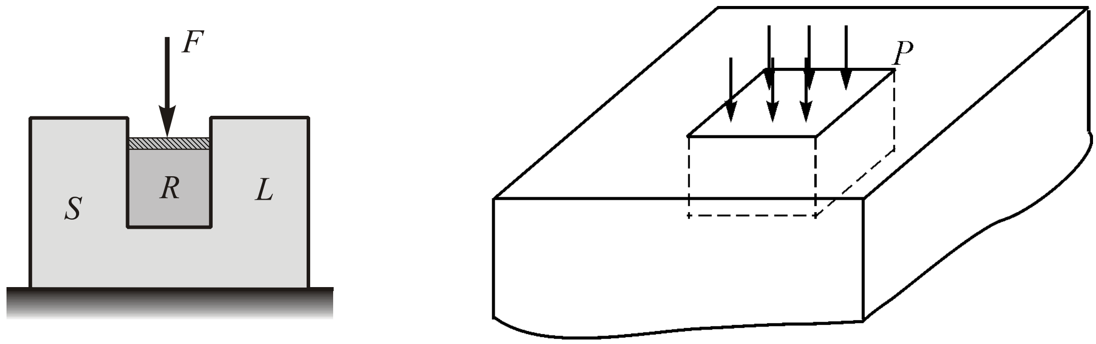


Fig. 22

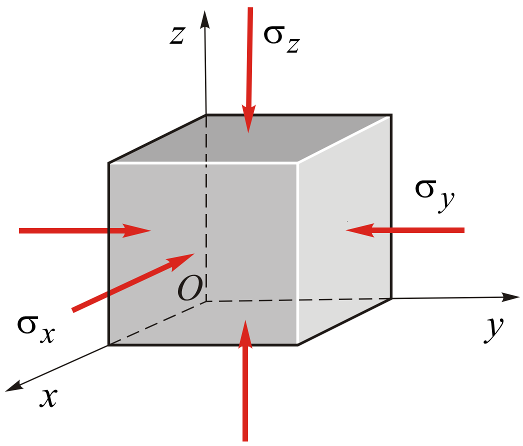


Fig. 23

Solution.

Disregarding friction between the block and array stress state of the R block may be described by Fig. 23. It is evident that $\sigma_z = -60\,\text{MPa}$, $\sigma_x = \sigma_y < 0$, $\varepsilon_x = \varepsilon_y = 0$.

Three unknown components of stress-strain state we will determine using the equations of generalized Hooke's law (19):

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \mu\left(\sigma_y + \sigma_z\right)\right],$$
$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \mu\left(\sigma_x + \sigma_z\right)\right],$$
$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \mu\left(\sigma_y + \sigma_x\right)\right],$$

or for our case (in MPa):

$$\begin{cases} \varepsilon_x = 0 = \dfrac{1}{70\times 10^3}\left[\sigma_x - 0.3\times\left(\sigma_y + (-60)\right)\right], \\ \varepsilon_y = 0 = \dfrac{1}{70\times 10^3}\left[\sigma_y - 0.3\times\left(\sigma_x + (-60)\right)\right], \\ \varepsilon_z = \dfrac{1}{70\times 10^3}\left[(-60) - 0.3\times\left(\sigma_x + \sigma_y\right)\right]. \end{cases}$$

In result of solution we get $\sigma_y = \sigma_x = -25.7\,\text{MPa}$, $\varepsilon_z = -64\times 10^{-3}$,

Unit volume change

$$e_V = \frac{V - V_0}{V_0} = \varepsilon_x + \varepsilon_y + \varepsilon_z = 0 + 0 - 64\times 10^{-3} = -64\times 10^{-3} \text{ (volume decreases).}$$

Strain energy density is determined by formula (43).

In our case of loading $U_0 = \dfrac{1}{2\times 70\times 10^3}\times$

$$\times\left[(-25.7)^2 + (-25.7)^2 + (-60)^2 - 2\times 0.3\left((-25.7)\times(-25.7) + (-25.7)\times(-60) + (-60)\times(-25.7)\right)\right] =$$

$$= 38\times 10^{-3}\ \text{G/m}^3.$$