LECTURE 12 Strength of a Bar in Torsion

A bar subjected to torsion is called a **shaft**. In torsion of a shaft there appears a single internal force factor, a **twisting moment (torque)**, which acts in the plane of the shaft cross-section. The examples of torsional deformation are shown in Figs 1–10.

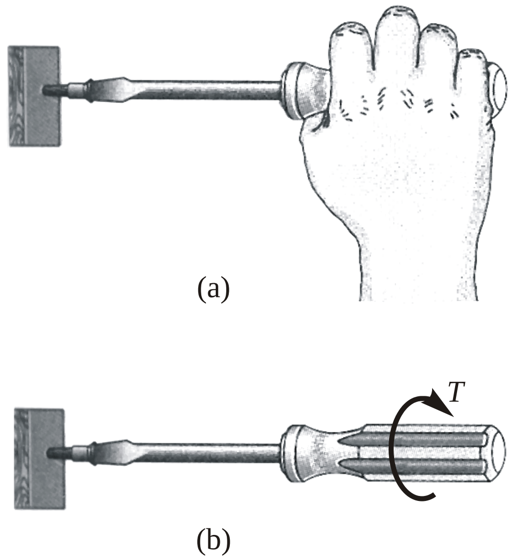


Fig. 1 Torsion of a screwdriver due to a torque *T* applied to the handle

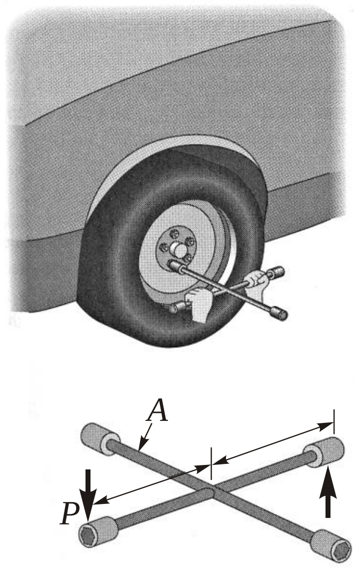


Fig. 2 A lug wrench

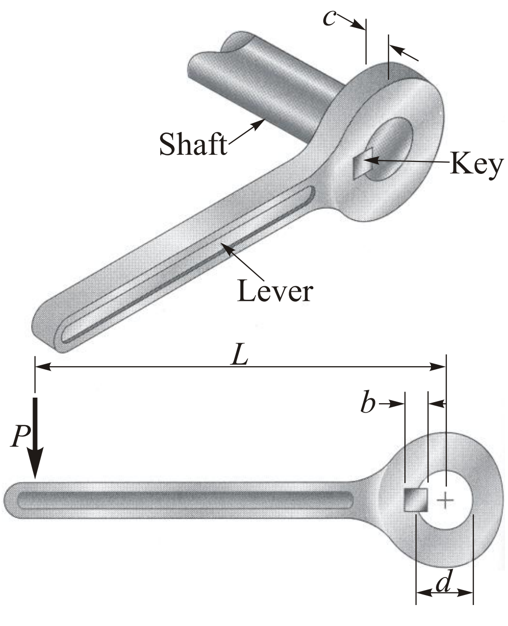


Fig. 3 A special lever is used to twist a circular shaft by means of a square key that fits into keyways in the shaft in lever

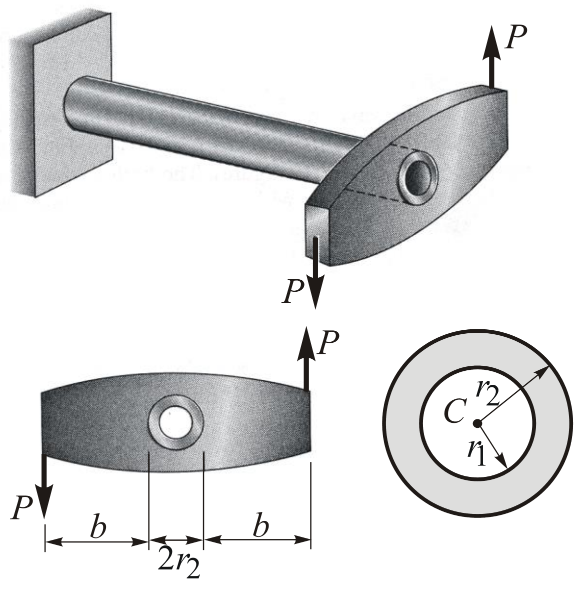


Fig. 4 Circular tube subjected to a torque produced by a forces *P*

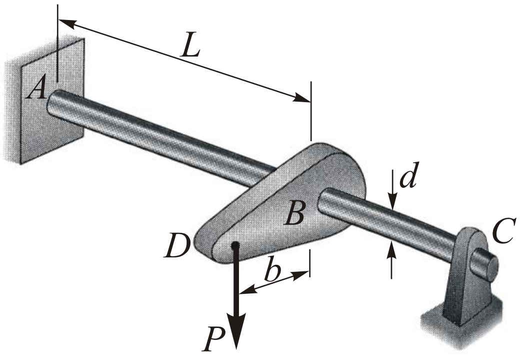


Fig. 5 Round solid bar loaded by a vertical force P

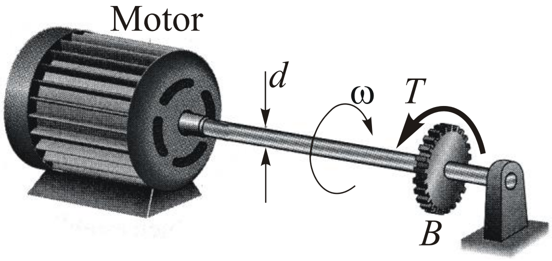


Fig. 6 A solid circular steel shaft transmits the torque from the motor to the gear

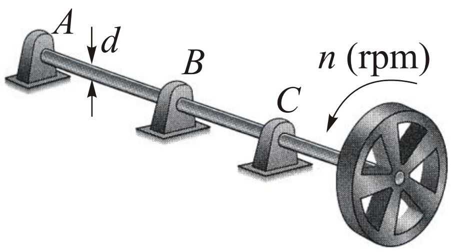


Fig. 7 Rotating flywheel creates torsional deformation of the shaft if the bearing at A suddenly freezes

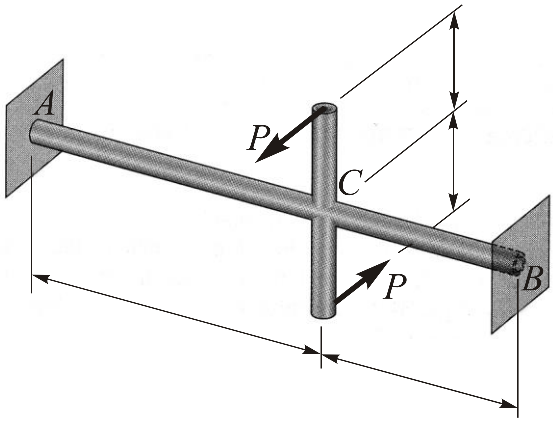


Fig. 8 A hollow steel shaft is held against rotation at ends A and B

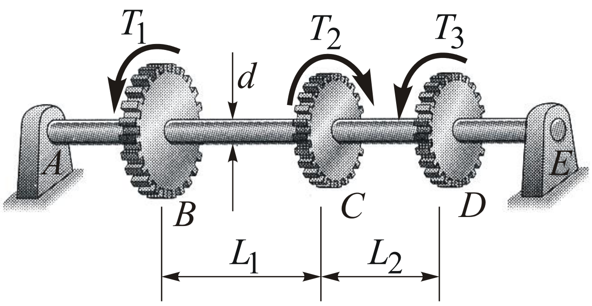


Fig. 9 Solid steel shaft turns freely in bearings at A and E points. It is driven by a gear at C point. T_1 and T_3 are resisting torques

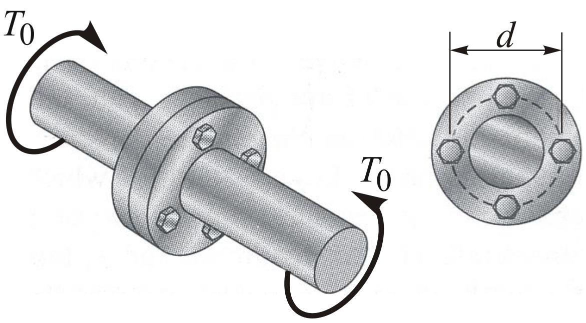


Fig. 10 A torque T_0 is transmitted between two flanged shafts

1 Correlation Between Shearing Stress and Twisting Moment

This can be done by considering the following experiment. Let a square network be applied onto the outside surface of a **round shaft** (see Fig. 11):

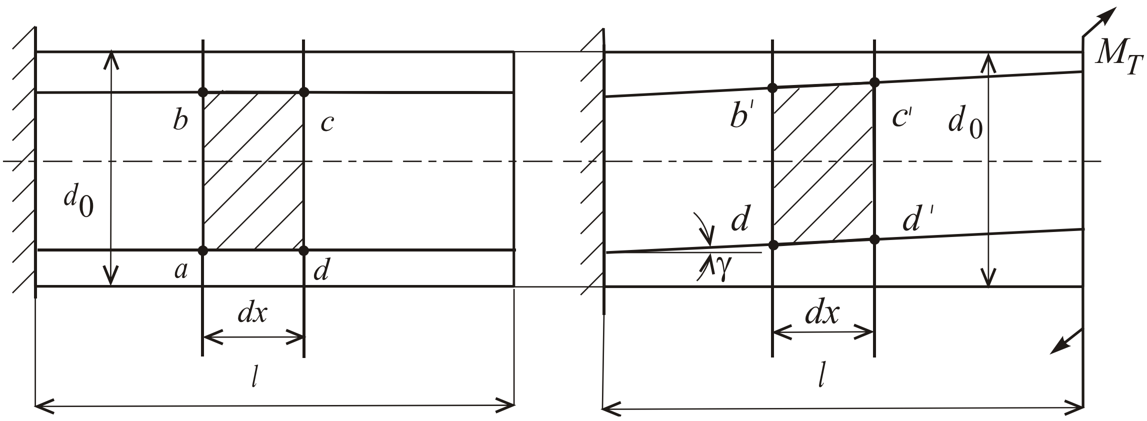


Fig. 11

Suppose that the shaft is rigidly fixed at one end and is twisted by external moment M_K at the other. The following picture will then be observed on the application of the moment:

(1) ***All sections of the shaft remain plane after torsion deformation*** and only turn relative to one another. The ***radius of the shaft cross-section has not changed*** in torsion.

(2) ***The distance dx and total length l of the shaft have not changed,*** since there are no longitudinal forces and stresses which could deform the shaft in the axial direction.

(3) The ***square network on the shaft surface changes to a rhombic***, i.e. each square deforms similarly to the shear deformation of the bar. Thus, ***there are only tangential (shearing) stresses which appear in the cross section*** of a shaft.

So, we have got **pure shear** deformation at any point of the shaft (see Fig. 12):

$$\tau = G\gamma . \qquad (1)$$

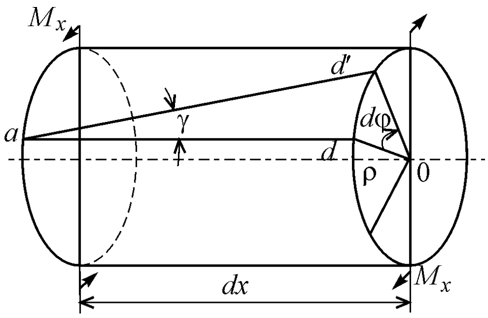


Fig. 12

In Fig. 12 which shows the equilibrium of infinitesimally in length segment of the shaft under internal torque moment M_x it is seen that

$$0 \le \rho \le \frac{D}{2}, \quad dd' = \gamma dx = \rho d\varphi,$$

hence

$$\gamma = \rho \frac{d\varphi}{dx} \tag{2}$$

where γ is an **angle of shear**; $d\varphi$ is an **elementary angle of twist**.

To go over to shear stresses, let us use **Hooke's law in shear** and substitute into it the expression for γ:

$$\tau = G\gamma = \rho \frac{d\varphi}{dx} G. \tag{3}$$

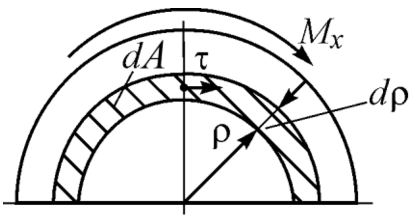


Fig. 13

If an elementary area dA at a distance ρ from the centroid of the section is acted upon by the stress τ, the elementary torque moment will be equal to the elementary force τdA multiplied by the polar radius ρ (Fig. 13):

$$dM_x = \tau\rho dA.$$

The total moment can be obtain by summing the elementary moments over the area of the section:

$$M_x = \int_A \tau\rho dA = G\frac{d\varphi}{dx}\int_A \rho^2 dA. \tag{4}$$

It may be recalled that the integral obtained

$$\int_A \rho^2 dA = I_\rho$$

is called the **polar moment of inertia**. The expression for M_x can then be rewritten as follows:

$$M_x = GI_\rho \frac{d\varphi}{dx},$$

whence we obtain

$$\frac{d\varphi}{dx} = \frac{M_x}{GI_\rho}. \tag{5}$$

According to expression (3)

$$\frac{d\varphi}{dx} = \frac{\tau}{\rho G}. \tag{6}$$

Consequently

$$\frac{\tau}{G\rho} = \frac{M_x}{GI_\rho} \quad \rightarrow \quad \tau(\rho) = \frac{M_x \rho}{I_\rho}. \tag{7}$$

As may be seen from this formula, ***shearing stresses are directly proportional to the radius of a point. They are distributed over the cross-section according to a linear law and have a maximum value at points most remote from the axis of the shaft***. Then

$$\tau_{\max} = \frac{M_x \rho_{\max}}{I_\rho}. \tag{8}$$

The quantity

$$\frac{I\rho}{\rho_{\max}} = W_\rho \,[\text{m}^3] \tag{9}$$

is called the **polar section(al) modulus**.

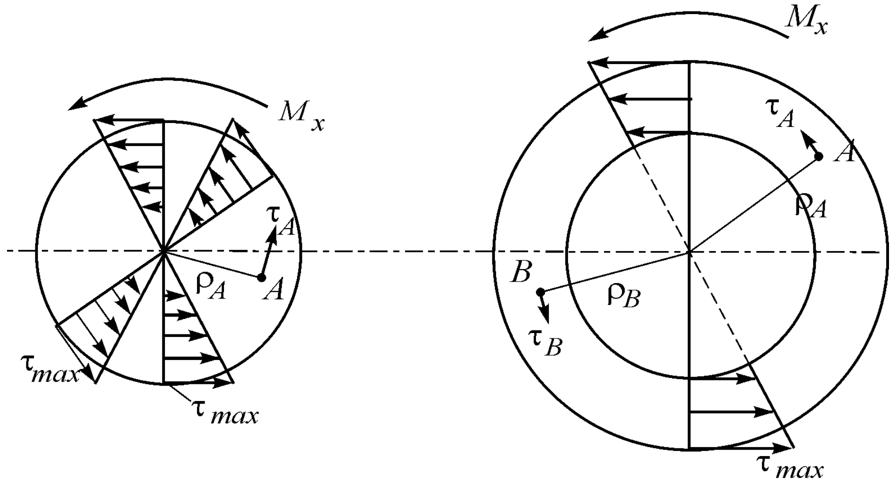
For circular sections of diameter *D*:

$$I_\rho^{\oslash} = \frac{\pi D^4}{32}, \quad W_\rho^{\oslash} = \frac{\pi D^4}{32} : \frac{D}{2} = \frac{\pi D^3}{16}, \tag{10}$$

and for a tubular section (hollow shaft):

$$I_\rho^{\circledcirc} = \left(I_\rho^{\oslash} - I_\rho^{\circ} \right) \Big/ \rho_{\max} = \left(\frac{\pi D^4}{32} - \frac{\pi d^4}{32} \right) \Big/ \frac{D}{2} = \frac{\pi D^3}{16} \left(1 - \alpha^4 \right); \quad \alpha = \frac{d}{D}. \qquad (11)$$

Let us draw the diagram of shearing stresses in the section of a shaft:



$$\tau_A = \frac{M_x \rho_A}{I_\rho} \qquad\qquad \tau_B = \frac{M_x \rho_B}{I_\rho}$$

Fig. 14

The maximum ***tangential stresses act all over the external surface*** of the shaft.

2 Condition of Strength for a Shaft in Torsion

The strength condition for a shaft in torsion states that the ***maximum working stress in torsion of a shaft in its critical section should be not higher than an allowable stress***:

$$\tau_{\max} = \frac{M_{x_{\max}}}{W_\rho} \le [\tau]. \qquad (12)$$

In rough calculations the allowable tangential stress can be determined by the formula:

$$[\tau] = 0.5[\sigma], \qquad (13)$$

Where $[\sigma]$ is the **allowable stress in tension** (to be taken from Handbooks or in result of testing).

The **condition of strength** makes it possible to solve three types of problems:

1) estimate (check) the strength of a shaft for specified load and dimensions using the condition:

$$\tau_{\max} \le [\tau]; \tag{14}$$

2) determine the required polar sectional modulus in torsion and diameter by the specified allowable stress $[\tau]$ and load M_x:

$$W_\rho \ge \frac{M_{x_{\max}}}{[\tau]}; \tag{15}$$

3) determine the allowable load on a shaft by the specified allowable stress and geometrical dimensions of the shaft section:

$$[M_x] = [\tau]\, W_\rho. \tag{16}$$

3 Determination of Angle of Twist

According to expression (5)

$$\frac{d\varphi}{dx} = \frac{M_x}{GI_\rho}, \tag{17}$$

whence

$$\varphi = \int_0^l \frac{M_x dx}{GI_\rho}, \tag{18}$$

where l is the distance between the sections for which the twisting angle φ is determined.

If the twisting moment does not vary along the length of the rod l and if the rigidity GI_ρ remains constant, then

$$\varphi = \frac{M_x l}{GI_\rho}. \tag{19}$$

For the stepped shaft

$$\varphi = \sum_{i=1}^{n} \varphi_i = \sum_{i=1}^{n} \frac{M_{x_i} l_i}{G_i I_{\rho_i}}. \tag{20}$$

4 Condition of Rigidity in Torsion

The **relative twisting angle** ψ can be introduced as follows:

$$\psi = \frac{\varphi}{l} = \frac{M_x}{GI_\rho}. \tag{21}$$

This angle is independent of the shaft length.

Shaft must satisfy the **condition of rigidity**, i.e. the ***maximum relative twisting angle*** ψ ***in the shaft critical segment must not exceed the allowable relative twisting angle*** $[\psi]$:

$$\psi_{\max} = \frac{M_{x_{\max}}}{GI_p} \le [\psi], \tag{22}$$

where $[\psi]$ is the allowable relative angle of twist in radians per meter of the shaft length.

The allowable relative twisting angle is usually specified in degrees per meter of length. The formula of shaft rigidity will then have a somewhat different form:

$$\psi_{\max} = \frac{180 M_{x_{\max}}}{\pi GI_p} \le [\psi]. \tag{23}$$

Like the condition of strength, the condition of rigidity makes it possible to solve similar three types of engineering problems.

5 Lines of Principal Stresses

The lines of principal stresses are **helical lines**. They are located at 45º angle to the shaft generatrix:

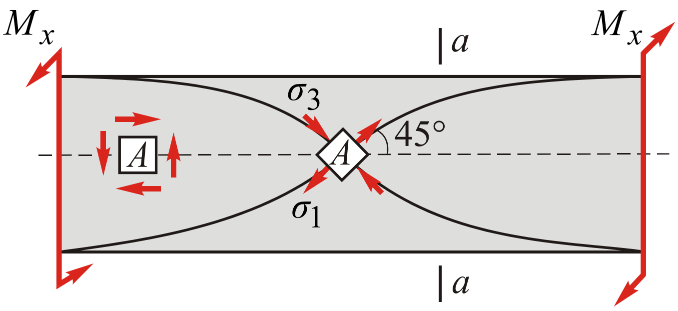


Fig. 15

6 Torsional Deformation of Rectangular Bar

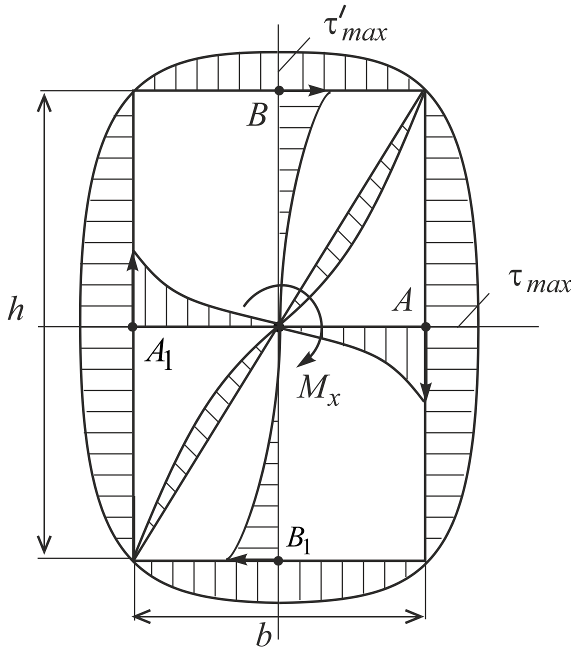


Fig. 16

Torsional deformation of bars of non-circular, in particular, of **rectangular cross section** is characterized in that the cross sections of the bar do not remain plane, but are curved beyond the cross-sectional plane.

As a result, the problem of stress calculation becomes more complicated than for round shafts, for which the hypothesis of plane sections is valid. The problem can be solved by the methods of the theory of elasticity.

The diagram of stresses is represented by the Fig. 16.

The ***maximum stresses occur at the middle of the long sides at the points A and A₁***:

$$\tau_{\max} = \tau_A = M_x / W_T \,, \tag{24}$$

where $W_T = \alpha h b^2$ is the **moment of resistance to torsion**; the factor α is a function $\alpha(h/b)$. Its numerical values are presented in the tables.

$$\tau'_{\max} = \gamma \tau_{\max} \,, \quad \gamma = \gamma(h/b). \tag{25}$$

The angular displacement is

$$\varphi = \frac{M_x l}{GI_T}, \qquad (26)$$

where $I_T = \beta h b^3$, $\beta = \beta(h/b)$.

The factors α, γ and β depend on the ratio of the sides h/b.

$\frac{h}{b}$	1	1.5	2.0	3.0	4.0	6.0	8.0	10.0	>10
α	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.313	0.333
γ	1	0.859	0.795	0.753	0.745	0.743	0.743	0.743	0.743

Example 1 Checking problem for solid uniform shaft

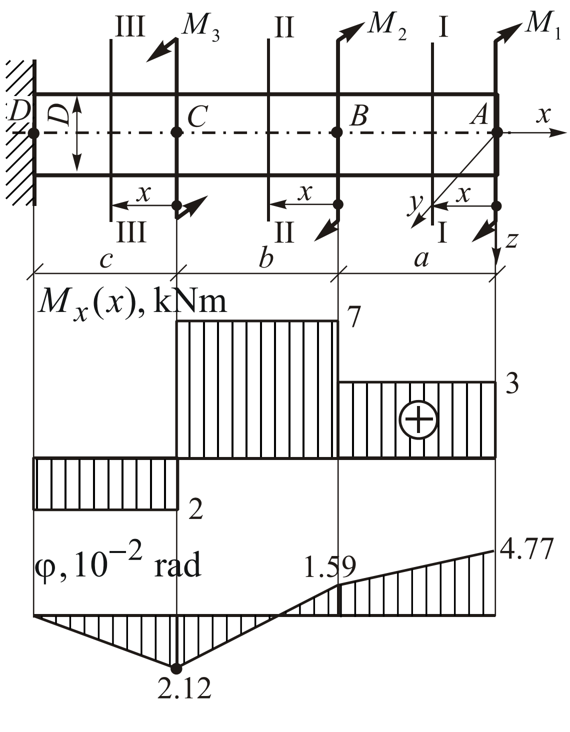


Fig. 17

Given: $M_1 = 3\,\text{kNm}$, $M_2 = 4\,\text{kNm}$, $M_3 = 9\,\text{kNm}$, $a = 2.0\,\text{m}$, $b = 1.0\,\text{m}$, $c = 2.0\,\text{m}$, $G = 8\times10^4\,\text{MPa}$, $D = 70\,\text{mm}$, $[\psi] = 1\,\text{degree/m}$, $[\tau] = 70\,\text{MPa}$.

R.D.: 1) determine internal torsional moments in the shaft cross-sections and draw the graph of their distribution;

2) check the strength and rigidity of the shaft;

3) design the graphs of twisting angle distribution.

Solution

1) Calculating internal torsional moments in the cross-sections applying method of sections (clockwise rotation is assumed to be positive).

I–I $0<x<2\,\text{m}$

$$M_x{}^{I}(x)=+M_1=+3\ \text{kNm};$$

II–II $0<x<1\,\text{m}$

$$M_x{}^{II}(x)=+M_1+M_2=3+4=7\ \text{kNm};$$

III–III $0<x<2\,\text{m}$

$$M_x{}^{III}(x)=+M_1+M_2-M_3=3+4-9=-2\,\text{kNm}.$$

Corresponding graph is shown on the Fig. 17. Second cross-section is critical.

$$\left|M_{x\max}\right|=7.0\,\text{kNm}.$$

2) Checking the strength of the shaft.

Calculating maximum acting stresses in critical points of critical cross-section:

$$\tau_{\max}=\frac{\left|M_{x\ \max}\right|}{W_\rho}=\frac{\left|16M_{x\ \max}\right|}{\pi D^3}=\frac{16\times7\times10^3}{3.14\left(70\times10^{-3}\right)^3}=104\,\text{MPa}.$$

Since $104>70$ MPa, the shaft is not strong.

3) Checking the rigidity of the shaft.

Determine maximal relative angle of twist.

$$\psi_{\max}=\frac{\left|M_{x\ \max}\right|}{GI_\rho};\quad I_\rho=\frac{\pi D^4}{32}=\frac{3.14\times\left(70\times10^{-3}\right)^4}{32}=235.6\times10^{-8}\,\text{m}^4.$$

$$\psi_{\max}=\frac{7\times10^3}{8\times10^{10}\times235.6\times10^{-8}}=3.71\times10^{-2}\,\text{rad/m}=$$

$$=3.71\times10^{-2}\times\frac{180^\circ}{\pi}=2.1\,\text{degree/m}.$$

Since, $\psi_{max}=2.1>1$ ($\psi_{max}>[\psi]$) the shaft is not rigid.

4) Designing the graph of twisting angle distribution.

For this, we will use the formula:

$$\varphi(x) = \frac{M_x x}{GI_\rho} = kx,$$

i.e. twisting angle formula really is linear function of the shaft length. It is necessary to calculate twisting angles of *C*, *B*, *A* cross-sections relative immobile *D* point and connect corresponding points of $\varphi(x)$ graph by straight lines.

In our case, torsional rigidity

$$GI_\rho = G\frac{\pi D^4}{32} = 8\times 10^{10} \times \frac{\pi}{32}\left(70\times 10^{-3}\right)^4 = 188480\,\text{Nm}.$$

Twisting angle of *C*-section will be determined by the formula:

$$\varphi_C = \varphi_{CD} = \frac{M_x{}^{III} \times c}{GI_\rho} = \frac{\left(-2\times 10^3\right)\times 2}{188480} = -2.12\times 10^{-2}\,\text{rad}.$$

Twisting angle of *B*-section relative to immobile *D*-point will be determined by the formula:

$$\varphi_B = \varphi_{BD} = \varphi_{CD} + \varphi_{BC} = -2.12\times 10^{-3} + \frac{M_x{}^{II} \times b}{GI_\rho} =$$

$$= -2.12\times 10^{-2} + \frac{(7\times 10^3)\times 1}{188480} = 1.59\times 10^{-2}\ \text{rad}.$$

Twisting angle of *A*-section relative to immobile *D*-point will be determined by the formula:

$$\varphi_A = \varphi_{AD} = \varphi_{BD} + \varphi_{AB} = 1.59\times 10^{-2} + \frac{M_x{}^{I} \times a}{GI_\rho} = 1.59\times 10^{-2} + \frac{3\times 10^3 \times 2}{188480} = 4.77\times 10^{-2}$$

rad.

Corresponding graph is shown on the Fig. 17

Example 2 Design problem for solid and hollow uniform shafts

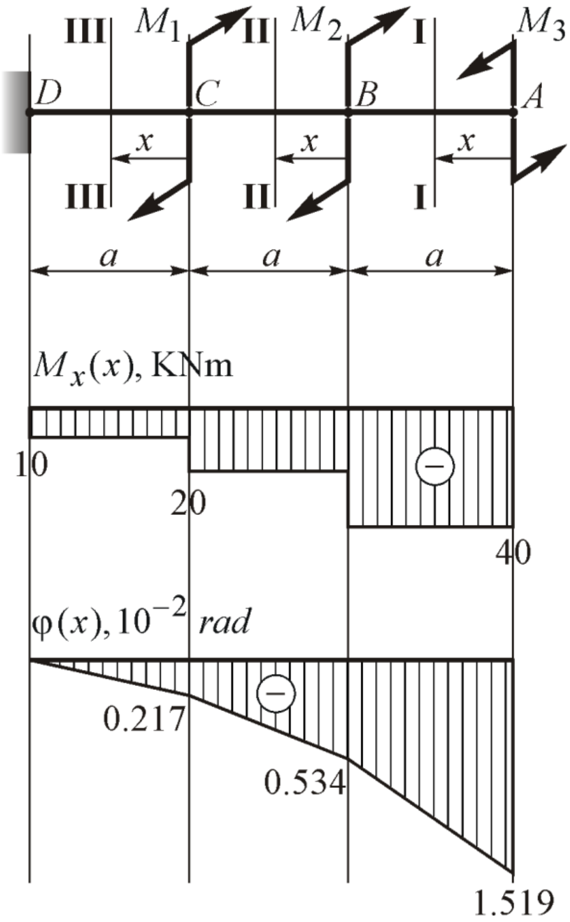


Fig. 18

Given: $M_1 = 10\,\text{kNm}$, $M_2 = 20\,\text{kNm}$, $M_3 = 40\,\text{kNm}$, $a = 0.5\,\text{m}$, $G = 8 \times 10^4\,\text{MPa}$, $[\psi] = 1\,\text{degree/m}$, $[\tau] = 100\,\text{MPa}$.

R.D.: 1) determine internal torsional moments in the shaft cross-sections and design the graph of their distribution (clockwise rotation is assumed to be positive);

2) determine the diameters of solid and hollow shafts from condition of strength taking into account thickness ratio $\alpha = d / D = 0.8$;

3) determine the diameters of solid and hollow shafts from condition of rigidity;

4) draw the graphs of stress distribution in critical cross-sections of the shafts designed;

5) compare the weights of solid and hollow strong shafts;

6) design the graphs of twisting angle distribution.

Solution

1) Determine internal torque moments in an arbitrary cross-sections of the shaft applying method of sections:

I–I $0 < x < a$

$$M_x^{I}(x) = -M_3 = -40\,\text{kNm};$$

II–II $0 < x < a$

$$M_x^{II}(x) = -M_3 + M_2 = -40 + 20 = -20\,\text{kNm};$$

III–III $0 < x < a$

$$M_x^{III}(x) = -M_3 + M_2 + M_1 = -40 + 20 + 10 = -10\,\text{kNm}.$$

In result, $\left|M_x(x)\right| = 40\,\text{kNm}$ and I–I section is critical.

2) Calculating the diameters of the shafts from the condition of strength.

a) for solid shaft:

$$\tau_{\max}^{\oslash} = \frac{\left|M_{x\max}\right|}{W_\rho} \le [\tau] \to \overset{\oslash}{D} \ge \sqrt[3]{\frac{16\left|M_{x\max}\right|}{\pi[\tau]}} = \sqrt[3]{\frac{16\times40\times10^3}{3.14\times100\times10^6}} = 0.127\,\text{m};$$

b) for hollow shaft in $\alpha = \dfrac{d}{D} = 0.8$:

$$\tau_{\max}^{\circledcirc} = \frac{\left|M_{x\max}\right|}{W_\rho\ (1-\alpha^4)} \le [\tau] \to \overset{\circledcirc}{D} \ge \sqrt[3]{\frac{16\left|M_{x\max}\right|}{\pi[\tau](1-\alpha^4)}} = \sqrt[3]{\frac{16\times40\times10^3}{3.14\times100\times10^6(1-0.8^4)}} = 0.151\text{m},$$

$$\overset{\circledcirc}{d} = 0.8\times0.151 = 0.121\text{m}.$$

3) Calculating the diameters of the shafts from the condition of rigidity.

a) for solid shaft:

$$\psi_{\max}^{\oslash} = \frac{\left|M_{x\max}\right|}{GI_\rho} \le [\psi]\frac{\pi}{180},\ \text{where } [\psi] \text{ – in degree/m.}$$

Then

$$\overset{\oslash}{I}_\rho = \frac{\pi D^4}{32} \ge \frac{\left|M_{x\max}\right|\times180}{\pi G[\psi]} = \frac{40\times10^3\times180}{3.14\times8\times10^4\times10^6} = 2.86\times10^{-5}\,\text{m}^4.$$

$$\overset{\oslash}{D} \ge \sqrt[4]{\frac{32I_\rho}{\pi}} = \sqrt[4]{\frac{32\times2.86\times10^{-5}}{3.14}} = 0.131\,\text{m}.$$

b) for hollow shaft:

$$\overset{\circledcirc}{I}_\rho = \frac{\pi \overset{\circledcirc}{D}{}^4}{32}(1-\alpha^4) = \frac{\left|M_{x\max}\right|\times180}{\pi G[\psi]} = 2.86\times10^{-5}\,\text{m}^4,$$

$$\overset{\circledcirc}{D} \ge \sqrt[4]{\frac{32\overset{\circledcirc}{I}_\rho}{\pi(1-\alpha^4)}} = \sqrt[4]{\frac{32\times2.86\times10^{-5}}{3.14(1-0.8^4)}} = 0.149\,\text{m}.$$

$\circledcirc$

$$d = 0.8 \times 0.149 = 0.119\,\text{m}.$$

After comparing the results we select for engineering application:

a) solid shaft with the diameter $\overset{\oslash}{D} = 0.131\text{m}$ as larger of two calculated values;

b) hollow shaft with the diameters

$$\overset{\circledcirc}{D} = 0.151\text{m}, \quad \overset{\circledcirc}{d} = 0.121\text{m},$$

as the larger pair of two calculated ones.

4) designing the graphs of stress distribution in critical cross-section for two designed shafts.

For solid shaft:

$$\overset{\oslash}{\tau}_{\max} = \frac{M_{x\max}}{W_\rho} = \frac{16M_{x\max}}{\pi D^3} = \frac{16 \times 40 \times 10^3}{3.14 \times (0.131)^3} = 90.67\,\text{MPa}.$$

Note, that $\tau_{\max}$ is less than allowable stress for the shaft material since its diameter was calculated from condition of rigidity, i.e. $\psi_{\max} = [\psi]$.

For hollow shaft:

$$\overset{\circledcirc}{\tau}_{\max} = \frac{M_{x\max}}{W_\rho} = \frac{16M_{x\max}}{\pi D^3(1-\alpha^4)} = \frac{16 \times 40 \times 10^3}{3.14 \times (0.151)^3(1-0.8^4)} = 100\,\text{MPa}.$$

Note, that $\tau_{\max}$ is equal to allowable stress for the shaft material since its diameters were calculated from condition of strength, i.e. $\tau_{\max} = [\tau]$.

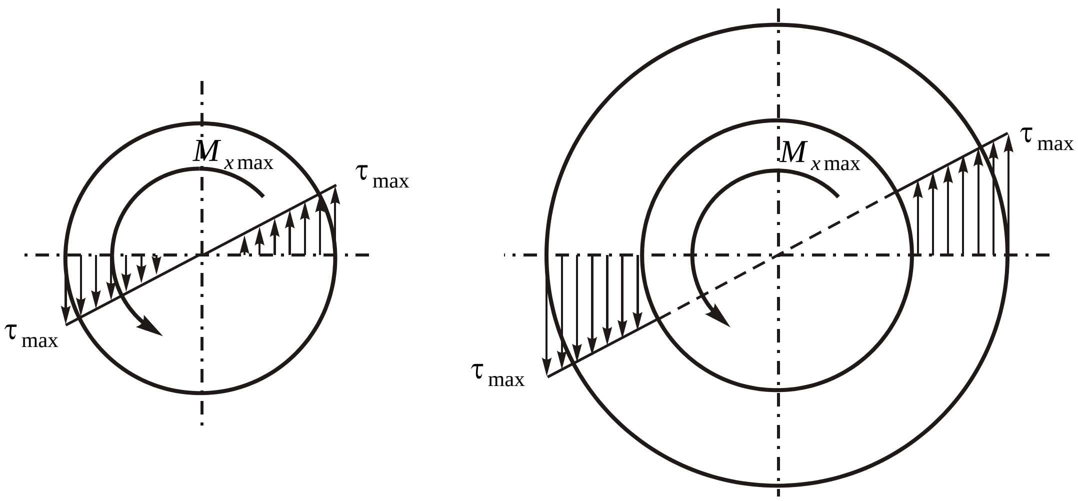


Fig. 18

5) Comparing the weights of strong solid and hollow shafts.

Fig. 18 shows that the core of solid shaft is under the action of low stresses, i.e. this part of cross-section is underloaded and can withstand larger stresses. This part of the cross-section material may be decreased if it will be situated as far as possible from the cross-section centroid. Taking into account this idea, hollow shaft must be more effective in weight to withstand the same torque moment.

Check this idea and find relationship of the weights of the same in length solid and hollow shafts.

$$\frac{G^{\oslash}}{G^{\circledcirc}}=\frac{\gamma V^{\oslash}}{\gamma V^{\circledcirc}}=\frac{S^{\oslash}\cdot 1\,\mathrm{m}}{S^{\circledcirc}\cdot 1\,\mathrm{m}}=\frac{4\pi D^{\oslash 2}}{4\left(\pi D^{\circledcirc 2}-\pi d^{\circledcirc 2}\right)}=\frac{D^{\oslash 2}}{D^{\circledcirc 2}-d^{\circledcirc 2}}.$$

Note, that to compare the weights, we will use the diameters determined from conditions of strength. Then

$$\frac{G^{\oslash}}{G^{\circledcirc}}=\frac{0.127^2}{0.151^2-0.121^2}=1.9.$$

Conclusion. Solid shaft is approximately 2 times greater in weight than hollow one due to more effective distribution of material in hollow shaft. This advantage will depend on the thickness ratio α: increase of α corresponds to decrease in the weight of hollow shaft.

6) Designing the graph of twisting angle distribution for solid shaft.

Note, that its diameter is $D^{\oslash}=0.131$m.

To design the graph, we will use the formula

$$\varphi(x)=\frac{M_x(x)}{GI_\rho}=kx.$$

Also, angles of twist for *C*, *B*, *A* cross-sections we will calculate relative to immobile *D*-section:

$$\varphi_C=\varphi_{CD}=\frac{M_x^{III}a}{GI_\rho}=-\frac{10\times10^3\times0.5\times32}{3.14(0.131)^4\times8\times10^{10}}=-0.217\times10^{-2}\,\text{rad};$$

$$\varphi_B=\varphi_{BD}=\varphi_{DC}+\varphi_{CB}=\varphi_{CD}+\frac{M_x^{II}a}{GI_\rho}=-0.217\times10^{-2}-\frac{20\times10^3\times0.5\times32}{3.14(0.131)^4\times8\times10^{10}}=$$

$$=-0.217\times10^{-2}-0.434\times10^{-2}=-0.651\times10^{-2}\,\text{rad};$$

$$\varphi_A=\varphi_{AD}=\varphi_{BD}+\varphi_{BA}=\varphi_{BD}+\frac{M_x^{I}a}{GI_\rho}=-0.651\times10^{-2}-\frac{40\times10^3\times0.5\times32}{3.14(0.131)^4\times8\times10^{10}}=$$

$$-0.651\times10^{-2}-0.868\times10^{-2}=-1.519\times10^{-2}\,\text{rad}.$$

Corresponding graph is shown in Fig. 18.

Example 3 Problem of allowable torque moment for hollow shaft

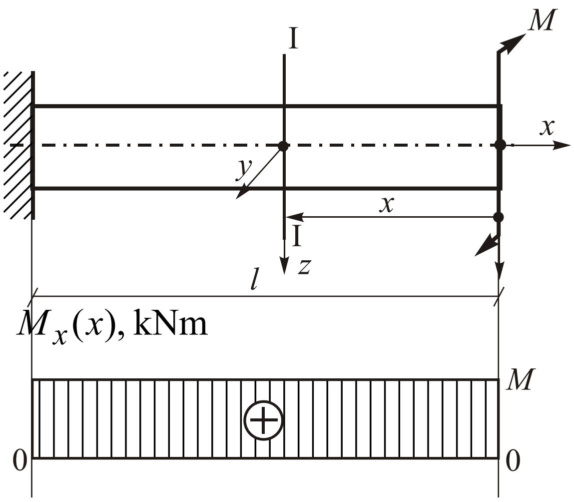


Fig. 19

Given: $[\tau]=80\,\text{MPa}$, $[\psi]=1\,\text{degree/m}$, $G=8\times10^4\,\text{MPa}$, hollow tube: $D=60\,\text{mm}$, thickness $t=5.0\,\text{mm}$, $l=3\,\text{m}$.

R.D.: allowable value of external torsional moment $[M]$.

Note. Since $[\tau]$ and $[\psi]$ are specified, it is necessary to calculate two values of allowable torque moment which correspond to condition of strength ($[M]_1$) and rigidity ($[M]_2$) and select lowest value.

1) Using method of sections, calculate internal torsional moment in an arbitrary cross-section I–I:

$$M_x{}^I(x)=+M\,.$$

It is evident that allowable value of internal torque moment will correlate with allowable value of external torque moment:

$$[M_x]=[M].$$

2) Calculating the allowable internal torque moment value $[M_x]_1$ from condition of strength.

Since $\tau_{\max} = \dfrac{M_x}{W_\rho} \le [\tau]$, then $[M_x]_1 = W_\rho[\tau]$; $W_\rho^{◎} = \dfrac{\pi D^3}{16}(1-\alpha^4)$;

$$\alpha = \frac{D-2t}{D} = 1 - \frac{2t}{D} = \frac{60-10}{60} = 0.83;$$

$$W_\rho^{◎} = \frac{3.14\times(60\times10^{-2})^3}{16}(1-0.83^4) = 22.25\times10^{-6}\,\text{m}^3.$$

After this, $[M_x]_1 = 22.25\times10^{-6}\times80\times10^6 = 1.78$ kNm,

Correspondent external allowable moment $[M]_1 = [M_x]_1 = 1.78$ kNm.

3) Calculating the allowable internal torque moment value $[M_x]_2$ from condition of rigidity.

Since $\psi_{\max} = \dfrac{M_{x\max}}{GI_\rho} \le [\psi]$, then $[M_x]_2 = GI_\rho[\varphi]$;

$$I_\rho^{◎} = \frac{\pi D^4}{32}(1-\alpha^4) = \frac{3.14\times(6\times10^{-2})^4}{32}(1-0.83^4) = 66.76\times10^{-8}\,\text{m}^4.$$

Thus, $[M_x]_2 = 8\times10^4\times10^6\times66.76\times10^{-8}\times1\times\dfrac{3.14}{180} = 0.932$ kNm.

Correspondent external allowable moment $[M]_2 = [M_x]_2 = 0.932$ kNm.

Larger of two moments, $[M]_1 = 1.78$ kNm does not satisfy condition of rigidity due to exceeded twisting angles. That is why, smaller of two moments is selected as solution of the problem: $[M] = [M]_2 = 0.932$ kNm.

Example 4 Problem of allowable torque moment for composite shaft

Given: combined round rectangle shaft. Round part: $d = 4\times10^{-2}$ m. Rectangle part: $h = 1.5b$, $b = d = 4\times10^{-2}$ m, $\tau = 100$ MPa.

R.D.: allowable value of external torsional moment [*M*].

Solution

Note, that in both parts of the shaft $|M_x| = |M|$ (counterclockwise rotation was assumed as negative).

1) calculating the allowable torsional moment from the condition of strength for solid round part:

$$\tau_{\max} = \frac{M_x}{W_\rho} \le [\tau] \rightarrow [M_x]_1 = W_\rho \times [\tau] = \frac{\pi D^3}{16}[\tau] =$$

$$= 100 \times 10^6 \times \frac{3.14 \times (4 \times 10^{-2})^3}{16} = 1.256\,\text{kNm}.$$

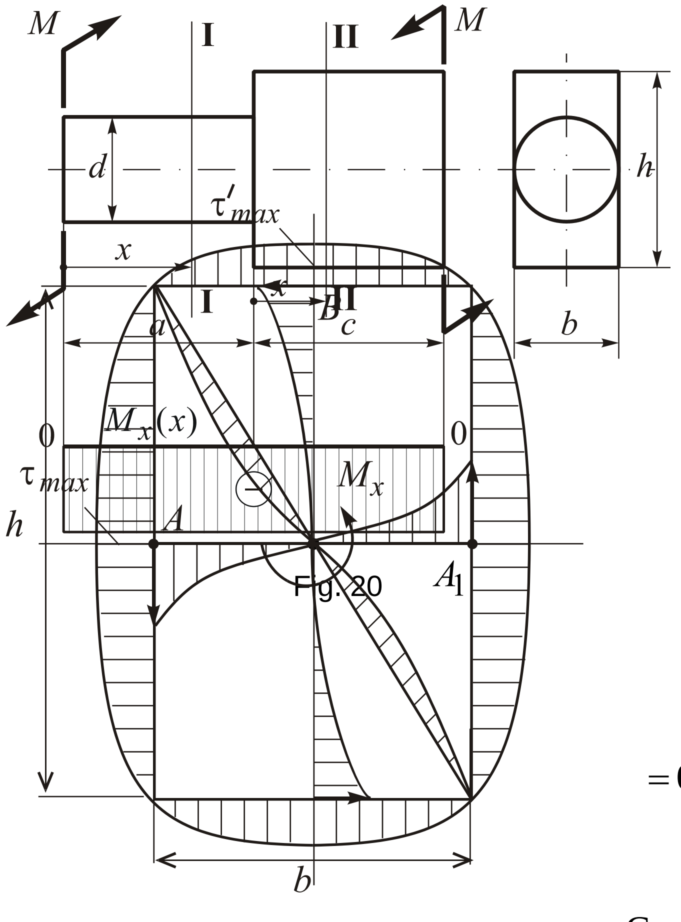


Fig. 20

Fig. 21

Corresponding external allowable torsional moment $[M]_1 = [M_x]_1 = 1.256\,\text{kNm}.$

2) calculating the allowable torsional moment from the condition of strength for rectangle part (midpoint A of large side is critical for this cross-section):

$$\tau_{\max} = \tau_A = \frac{M_x}{W_t} \le [\tau], \quad W_t = \alpha h b^2.$$

For $h/b = 1.5 \quad \alpha = 0.231.$

$$[M_x]_2 = W_t[\tau] = \alpha h b^2 [\tau] =$$

$$= 0.231 \times (4 \times 10^{-2})^2 \times 6 \times 10^{-2} \times 100 \times 10^6 =$$

$$= 2.218\,\text{kNm}.$$

Corresponding external allowable torsional moment

$$[M]_2 = [M_x]_2 = 2.218\,\text{kNm}.$$

Actual allowable torsional moment is less of two calculated ones: $[M] = 1.256\,\text{kNm}.$ Under this and less loading, all cross-sections of the shaft will be strong.

Example 5 Checking problem for statically indeterminate solid round shaft

Given: solid round shaft of $D = 50\,\text{mm}$ is loaded by concentrated moment $M = 10\,\text{kNm}$, $G = 8\times10^4\,\text{MPa}$, $[\tau] = 70\,\text{MPa}$, $[\psi] = 2\,\text{degree/m}$. $a = 2\,\text{m}$, $b = 3\,\text{m}$.

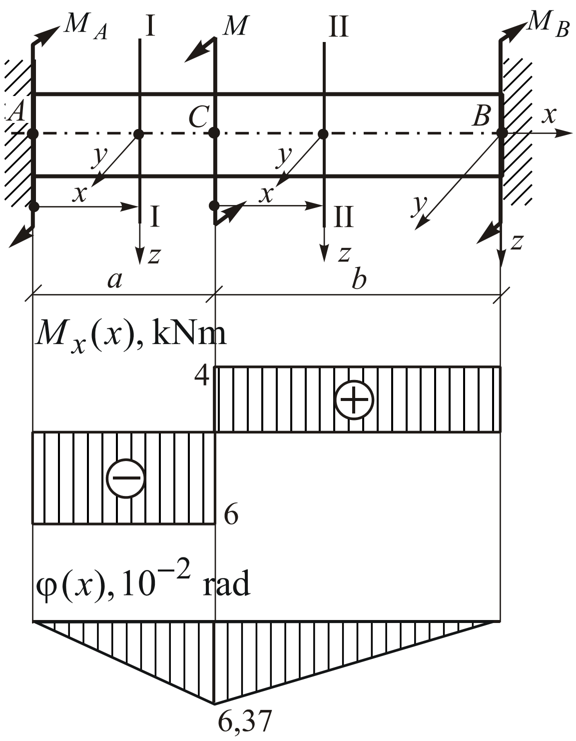


Fig. 22

R.D.: Check the strength and rigidity of the shaft and design the graph of twisting angles.

Note. Strength and rigidity analysis is impossible without preliminary calculation of reactive moments M_A and M_B.

Solution

1) Calculating the reactive moments in supports M_A and M_B.

a) Equation of equilibrium is

$$\sum M_x = M_B + M_A - M = 0.$$

Note, that clockwise rotation was assumed to be positive.

Since there are two unknown values and only one equation it is necessary to find complementary equation. It is evident, that all cross-sections are rotated relative to each other, but two cross-sections A and B are immobile. This facts allows creating complementary equation as **equation of angular deformations compatibility**

$$\varphi_{AB} = 0.$$

It is evident that $\varphi_{AB} = \varphi_{AC} + \varphi_{CB} = 0$ or

$$\varphi_{AC} = \frac{M^I{}_x(x)a}{GI_\rho}, \quad \varphi_{AC} = \frac{M^{II}{}_x(x)b}{GI_\rho}.$$

Since $M^I{}_x(x) = -M_A$; $M^{II}{}_x(x) = -M_A + M$,

$$\varphi_{AB} = \frac{-M_A}{GI_\rho} + \frac{(-M_A + M)b}{GI_\rho} = 0.$$

In result, $M_A = \frac{Mb}{a+b} = \frac{10\times10^3\times3}{(2+3)} = 6\,\text{kNm}$, and ***static indeterminacy is opened***.

2) Equations of internal torsional moments and corresponding graph $M_x(x)$.

I–I $0 < x < 2\,\text{m}$

$M_x{}^I(x) = -M_A = -6\,\text{kNm}$,

II–II $0 < x < 3\,\text{m}$

$M_x{}^{II}(x) = -M_A + M = -6 + 10 = +4\,\text{kNm}$.

Graph $M_x(x)$ is shown in Fig. 22. Note, that the abrupt in *C* cross-section is equal to external moment $M = 10\,\text{kNm}$, which is applied in this cross-section.

3) Checking the shaft strength.

Critical cross-section is determined by the moment

$$|M_{x\max}| = \left|M_x{}^I(x)\right| = 6\,\text{kNm}.$$

Calculating the maximum acting stress:

$$\tau_{\max} = \frac{|M_{x\max}|}{W_\rho} = \frac{|16M_{x\max}|}{\pi D^3} = \frac{16\times6\times10^3}{3.14(70\times10^{-3})^3} = 89\,\text{MPa}.$$

Since 89 MPa > 70 MPa ($\tau_{\max} > [\tau]$), ***the shaft is not strong***.

4) Checking the shaft rigidity.

Critical cross-section is determined by the same moment $|M_{x\max}| = 6\,\text{kNm}$.

Calculate relative twisting angle in critical portion of the shaft:

$$\psi_{\max} = \frac{|M_{x\max}|}{GI_\rho} = \frac{32|M_{x\max}|180}{\pi^2 GD^4} = \frac{32\times6\times10^3\times180}{3.14\times8\times10^4\times10^6\times(70\times10^{-3})^4} = 1.82\,\text{degree/m}$$

Since 1.82 degree/m < 2 degree/m ($\psi_{\max} < [\psi]$), ***the shaft is rigid***.

5) Design of the twisting angles graph.

Note, that left support *A* is selected as origin.

Then

$$\varphi_C = \varphi_{AC} = \frac{M_x{}^I(x)a}{GI_\rho} = \frac{32 M_x{}^I(x)a}{G\pi D^4} =$$

$$= \frac{32(-6\times10^3)\times 2}{8\times10^4\times10^6\times3.14\times(7\times10^{-2})^4} = -6.37\times10^{-2}\,\text{rad},$$

$$\varphi_B = \varphi_{AB} = \varphi_{AC} + \varphi_{CB} = -6.37\times10^{-2} + \frac{M_x{}^{II}(x)b}{GI_\rho} = -6.37\times10^{-2} + \frac{32 M_x{}^{II}(x)b}{G\pi D^4} =$$

$$= -6.37\times10^{-2} + \frac{32\times4\times10^3\times3}{8\times10^4\times10^6\times3.14\times(7\times10^{-2})^4} = -6.37\times10^{-2} + 6.37\times10^{-2} = 0\,.$$

Note, that this result may be predicted since B cross-section is rigidly fixed. The $\varphi(x)$ graph is shown in Fig. 22.

Example 6 Design problem for statically indeterminate solid and hollow nonuniform shafts

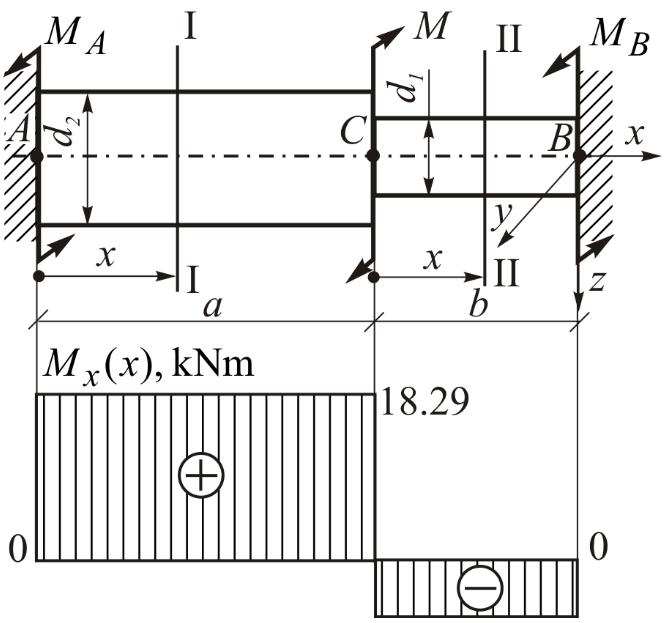


Fig. 23

Given: round stepped steel shaft is rigidly fixed in A and B points and loaded by external torsional moment $M = 20\,\text{kNm}$. Also, $[\tau] = 100\,\text{MPa}$, $[\psi] = 1\,\text{degree/m}$, $G = 8\times10^4\,\text{MPa}$, $a = 0{,}6\,\text{m}$, $b = 0{,}4\,\text{m}$, $d_2/d_1 = 2$.

R.D.: calculate d_1 and d_2 from conditions of strength and rigidity.

Solution

Note, that calculation of internal torsional moments is impossible without preliminary calculating the reactive moments M_A and M_B.

1) Calculating the reactive moments M_A and M_B.

a) from equation of equilibrium $\sum M_x = 0$:

$$\sum M_x = 0 = M_A + M_B - M\,.$$

It is evident that two unknown values M_A and M_B can not be determined from one equation. Complementary compatibility equation is:

b) $$\varphi_{AB} = 0 = \varphi_{AC} + \varphi_{CB}.$$

It may be rewritten taking into account that $\varphi_{AC} = \dfrac{M_x{}^I(x)a}{GI_\rho{}^I}$; $\varphi_{CB} = \dfrac{M_x{}^{II}(x)b}{GI_\rho{}^{II}}$.

Applying method of sections it becomes clear that

$$M_x{}^I(x) = +M_A;\ M_x{}^{II}(x) = M_A - M.$$

Note, that $I_\rho{}^I = \dfrac{\pi d_2{}^4}{32}$ and $I_\rho{}^{II} = \dfrac{\pi d_1{}^4}{32}$ are unknown values but $d_2 / d_1 = 2$ and $I_\rho{}^I / I_\rho{}^{II} = 16$.

After this finding we have that: $\dfrac{M_A a}{GI_\rho{}^I} + \dfrac{(M_A - M)b}{GI_\rho{}^{II}} = 0$ or

$M_A a + 16(M_A - M)b = 0$;

$$M_A = 18.29\,\text{kNm},$$

$$M_B = M - M_A = 1.71\,\text{kNm}.$$

2) Designing the graph $M_x(x)$.

I–I $0 < x < a$

$M_x{}^I(x) = M_A = +18.29\,\text{kNm}$;

II–II $0 < x < b$

$M_x{}^{II}(x) = M_A - M = -1.71\,\text{kNm}$. The graph $M_x(x)$ is shown in Fig. 22.

3) Calculating d_1 and d_2 from condition of strength.

Since these diameters are unknown it is not clear what portion is critical from the viewpoint of strength. But the relation between maximum stresses allows determining the critical portion:

$$\frac{\tau_{\max}^{I}}{\tau_{\max}^{II}}=\frac{M_x^{I}(x)W_\rho^{II}}{W_\rho^{I}M_x^{II}(x)}=\frac{M_x^{I}(x)}{M_x^{II}(x)}\frac{d_1^3}{d_2^3}=\frac{18.29}{1.71\times 8}=1.34\,.$$

Since $\tau_{\max}^{I}>\tau_{\max}^{II}$, first portion will be critical from the viewpoint of strength and condition of strength becomes

$$\tau_{\max}=\tau_{\max}^{I}=\frac{M_x^{I}(x)}{W_\rho^{I}}\le[\tau],\text{ and}$$

$$d_2\ge\sqrt[3]{\frac{16M_x^{I}(x)}{\pi[\tau]}}=\sqrt[3]{\frac{16\times 18.29\times 10^3}{3.14\times 100\times 10^6}}=9.77\times 10^{-2}\,\text{m.}$$

$$d_1=\frac{d_2}{2}=4.89\times 10^{-2}\,\text{m.}$$

4) Calculating d_1 i d_2 from condition of rigidity. To find actually critical portion let us calculate relation between relative twisting angles:

$$\frac{\psi^{II}}{\psi^{I}}=\frac{M_x^{II}(x)}{GI_\rho^{II}}\frac{GI_\rho^{I}}{M_x^{I}(x)}=\frac{M_x^{II}(x)}{M_x^{I}(x)}\frac{d_2^4}{d_1^4}=\frac{1.71\times 16}{18.29}=1.5\,.$$

Since $\psi^{II}>\psi^{I}$, the second portion will be critical from the viewpoint of rigidity. Corresponding condition of rigidity is: $\psi_{\max}=\psi^{II}=\dfrac{M_x^{II}(x)}{GI_\rho^{II}}\le[\psi]$, and

$$d_1\ge\sqrt[4]{\frac{32M_x^{II}(x)}{\pi G[\psi]}\times\frac{180}{\pi}}=\sqrt[4]{\frac{32\times 1.71\times 10^3\times 180}{3.14^2\times 8\times 10^4\times 10^6\times 1}}=5.94\times 10^{-2}\,\text{m.}$$

$$d_2=2d_1=11.8\times 10^{-2}\,\text{m.}$$

Final selection is $d_1=5.94\times 10^{-2}$ m and $d_2=11.8\times 10^{-2}$ m. This pair satisfies both conditions of strength and rigidity.

MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National aerospace university “Kharkiv Aviation Institute”

Department of aircraft strength

Course
Mechanics of materials and structures

HOME PROBLEM 8

Strength and Rigidity Analysis of Statically Determinate Shaft

Name of student:

Group:

Advisor:

Data of submission:

Mark:

National aerospace university
"Kharkiv Aviation Institute"
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: Strength and Rigidity Analysis of Staticaly Determinate Shafts
Full name of the student, group

Variant: 1 **Complexity:** 1

Given: thickness ratio $\alpha = d/D = 0.8$; $[\tau] = 100$ MPa; $[\psi] = 1$ degree/m; $G = 8 \cdot 10^{10}$ Pa; $M_1 = 10$ kNm; $M_2 = 70$ kNm; $M_3 = 40$ kNm; $a = 2$ m; $b = 3$ m; $c = 4$ m.

Goal:
1) copy from home problem No3 graph of torsional moment distribution;
2) calculate the diameters of solid and hollow shafts using conditions of strength and rigidity;
3) draw the graphs of stress distributions in critical sections of solid and hollow shafts;
4) estimate the type of stress state in an arbitrary point of critical cross-section (selecting yourself solid or hollow shape of a shaft);
5) compare the weights of 1 meter-in-length solid and hollow strong shafts;
6) design the graph of twisting angle distribution for solid or hollow strong shaft (select youself).

Full name of the lecturer **signature**

Mark:

Design problem
for solid and hollow uniform shafts

Goal:
(1) to determine internal torque moment in the shaft cross-sections and to design the graph of it's distribution (clockwise rotation is assumed to be positive);
(2) calculate the diameter of solid shaft satisfying conditions of strength and rigidity;
(3) calculate the diameters of hollow shaft satisfying conditions of strength and rigidity, taking into account thickness ratio $\alpha = d/D = 0.8$;
(4) draw the graphs of stress distribution in critical cross-sections of both shafts;
(5) determine the type of stress state in an arbitrary point in critical cross-section of solid shaft;
(6) compare the weights of one meter-in-length solid and hollow strong shafts;
(7) calculate the angles of rotation for solid shaft and design the graph of twisting angle distribution.

Solution

1. Determine internal torque moments in an arbitrary cross-sections of the shaft applying the method of sections. Corresponding sign conventions are shown on Fig. 1. The shaft is shown on Fig. 2.

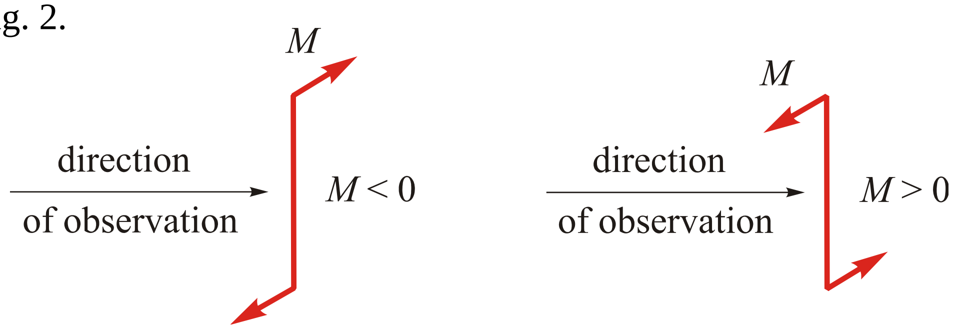


Fig. 1

I-I $(0 < x < c)$ $M_x^I(x) = M_3 = 40$ kNm,

II-II $(0 < x < b)$ $M_x^{II}(x) = M_3 - M_2 = 40 - 70 = -30$ kNm,

III-III $(0 < x < a)$ $M_x^{III}(x) = M_3 - M_2 + M_1 = 40 - 70 + 10 = -20$ kNm.

The graph $M_x(x)$ is shown on Fig. 2.

In result, $|M_x(x)| = 40$ kNm and I–I section is critical.

Note, that the graph $M_x(x)$ demonstrates that $M_x^{III}(x) = -20$ kNm. It allows determining the value of unknown moment M_0. Actually it has opposite direction of rotation, which is shown on Fig. 2 according to Fig. 1.

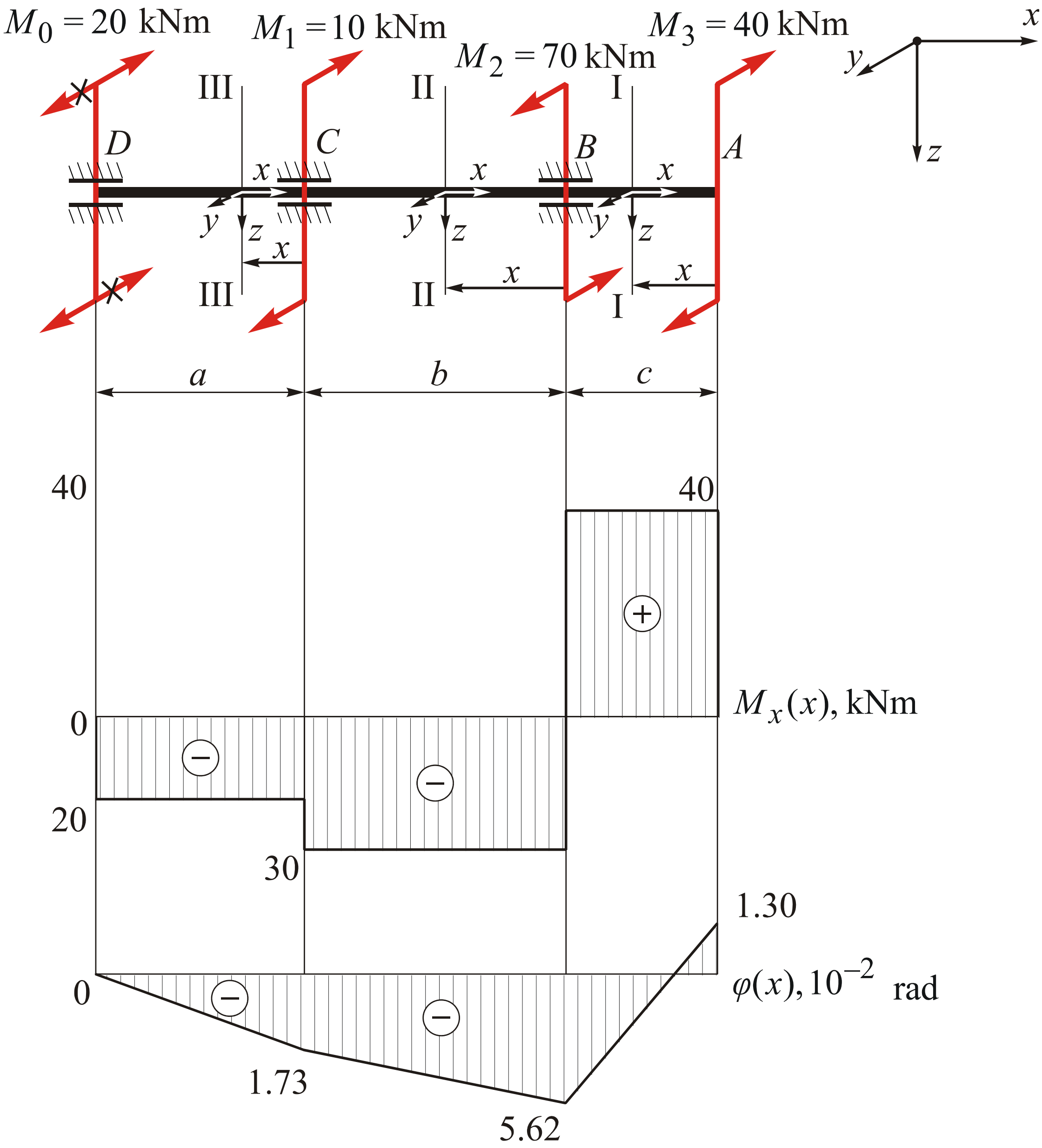


Fig. 2

2. Calculating the diameter of solid shaft:

2.1 satisfying the condition of strength:

$$\tau_{\max} = \frac{|M_{x_{\max}}|}{W_\rho} \le [\tau] \to D \ge \sqrt[3]{\frac{16|M_{x_{\max}}|}{\pi[\tau]}} = \sqrt[3]{\frac{16 \times 40 \times 10^3}{3{,}14 \times 100 \times 10^6}} = 0{,}127 \text{ m.}$$

2.2 satisfying the condition of rigidity:

$$\psi_{\max}=\frac{\left|M_{x_{\max}}\right|}{GI_{\rho}}\le[\psi]\rightarrow D^{\oslash}\ge\sqrt[4]{\frac{32\left|M_{x_{\max}}\right|}{\pi G[\psi]}\cdot\frac{180}{\pi}}=\sqrt[4]{\frac{32\times 40\times 10^{3}\times 180}{3{,}14^{2}\times 8\times 10^{10}\times 1}}=0{,}131\ \text{m}.$$

Finally, the solid shaft diameter is $D^{\oslash}=0{,}131$ **m. Note, it was calculated satisfying the condition of rigidity, i.e.** $\psi_{\max}=[\psi]$**.**

3. Calculating the diameters of hollow shaft in $\alpha=d/D=0{,}8$:

3.1 satisfying the condition of strength:

$$\tau^{\circledcirc}=\frac{\left|M_{x_{\max}}\right|}{W_{\rho}^{\circledcirc}}\le[\tau],$$

$$W_{\rho}^{\circledcirc}=\frac{\pi D^{\circledcirc 3}}{16}\left(1-\alpha^{4}\right)\rightarrow D^{\circledcirc}=\sqrt[3]{\frac{16\left|M_{max}\right|}{\pi[\tau](1-\alpha^{4})}}=\sqrt[3]{\frac{16\times 40\times 10^{3}}{3{,}14\times 100\times 10^{6}\times 0{,}5904}}=0{,}151\ \text{m},$$

$$d^{\circledcirc}=0{,}8D^{\circledcirc}=0{,}121\ \text{m}.$$

3.2 satisfying the condition of rigidity:

$$\psi_{\max}=\frac{\left|M_{max}\right|}{GI_{\rho}^{\circledcirc}}\le[\psi]\rightarrow I_{\rho}^{\circledcirc}=\frac{\pi D^{4}}{32}\left(1-\alpha^{4}\right)\rightarrow D^{\circledcirc}=\sqrt[4]{\frac{32\left|M_{max}\right|}{\pi G[\psi](1-\alpha^{4})}\times\frac{180}{\pi}}=$$

$$=\sqrt[4]{\frac{32\times 50\times 10^{3}\times 180}{3{,}14^{2}\times 8\times 10^{10}\times 1\times 0{,}5904}}=0{,}149\ \text{m},\quad d^{\circledcirc}=0{,}8D^{\circledcirc}=0{,}119\ \text{m}.$$

Finally, the hollow shaft diameters are $D^{\circledcirc}=0{,}151$ **m and** $d^{\circledcirc}=0{,}8D^{\circledcirc}=0{,}121$ **м. Note, they were calculated satisfying the condition of strength, i.e.** $\tau_{\max}=[\tau]$**.**

4. Drawing the graphs of stress distribution in critical cross-section for two designed shafts.

4.1 for solid shaft:

$$\tau_{\max}=\frac{M_{x\max}}{W_{\rho}}=\frac{16M_{x\max}}{\pi D^{\oslash 3}}=\frac{16\times 40\times 10^{3}}{3.14\times(0.131)^{3}}=90.7\ \text{MPa}.$$

Note, that $\tau_{\max}$ **is less than allowable stress for the shaft material since its diameter was calculated satisfying condition of rigidity, i.e.** $\psi_{\max}=[\psi]$**.**

4.2 for hollow shaft:

$$\tau_{\max}=\frac{M_{x\max}}{W_{\rho}}=\frac{16M_{x\max}}{\pi D^{\circledcirc 3}(1-\alpha^{4})}=\frac{16\times 40\times 10^{3}}{3.14\times(0.151)^{3}(1-0.8^{4})}=100\ \text{MPa}.$$

Note, that $\tau_{\max}$ **is equal to allowable stress for the shaft material since its diameters were calculated satisfying condition of strength, i.e.** $\tau_{\max}=[\tau]$**.**

The graphs of stress distribution are shown on Fig. 3.

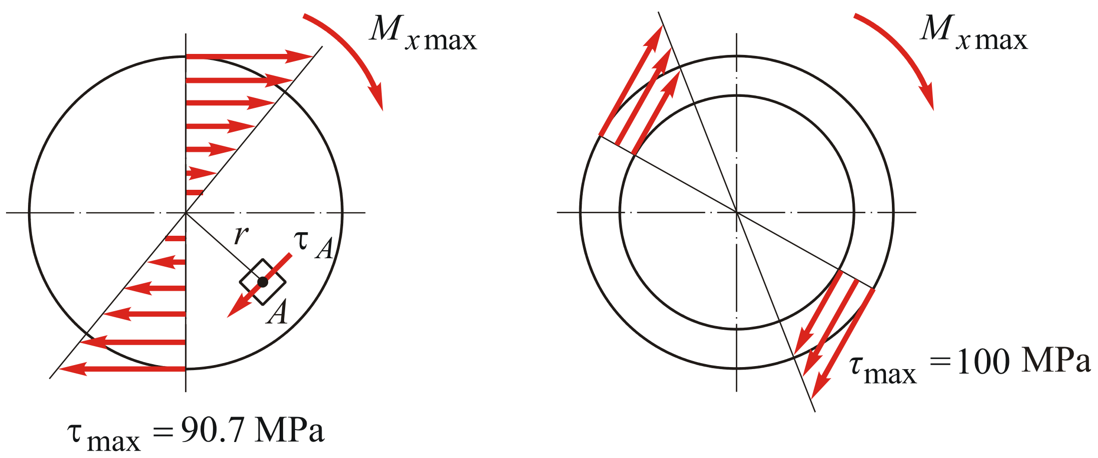


Fig. 3

5. Analysis of the stress state type in an arbitrary point A of critical section of solid shaft with polar coordinate $r = D/10$.

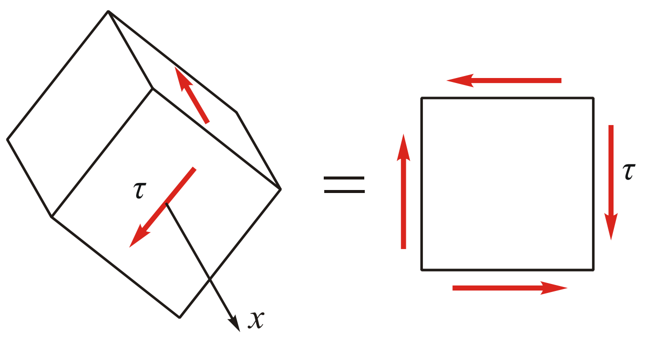


Fig. 3

$$\tau = \frac{M_{x\max}}{I_\rho} \times \frac{D}{10} = \frac{32M_{x\max}}{\pi D^4} \times \frac{D}{10} = \frac{3.2M_{x\max}}{\pi D^3} = \frac{3.2 \times 40 \times 10^3}{3.14 \times 0.131^3} = 18.1\,\text{MPa}.$$

Conclusion: stress state is biaxial and deformation is pure shear.

6. Comparing the weights of one meter-in-length segments of strong solid and hollow shafts.

The left graph of stress distribution shows that the core of solid shaft is under the action of low stresses, i.e. this part of cross-section is underloaded and can potentially withstand larger stresses. This part of the cross-section material may be used more effectively if it will be situated as far as possible from the cross-section centroid. Taking into consideration this idea, hollow shaft must be more effective in weight to withstand the same torque moment.

Check this idea and find relationship of the weights of the same in length solid and hollow shafts:

$$\frac{G^{\oslash}}{G^{\circledcirc}} = \frac{\gamma V^{\oslash}}{\gamma V^{\circledcirc}} = \frac{A^{\oslash} \times 1\,\text{m}}{A^{\circledcirc} \times 1\,\text{m}} = \frac{4\pi D^{\oslash 2}}{4\left(\pi D^{\circledcirc 2} - \pi d^{\circledcirc 2}\right)} = \frac{D^{\oslash 2}}{D^{\circledcirc 2} - d^{\circledcirc 2}}.$$

Note, that to compare the weights, we will use the diameters which were determined from conditions of strength. Then

$$\frac{G^{\oslash}}{G^{\circledcirc}} = \frac{0.127^2}{0.151^2 - 0.121^2} = 1.97\,.$$

Conclusion. Solid shaft is approximately 2 times greater in weight than hollow one due to more effective distribution of material in hollow shaft. This advantage will depend on the thickness ratio α: increase of α leads to decrease of hollow shaft weight.

7. Designing the graph of twisting angle distribution for solid shaft.
Note, that its diameter is $D^{\oslash} = 0.131$ m.
To design the graph, we will use the formula

$$\varphi(x) = \frac{M_x(x)}{GI_\rho} = kx \text{ – linear function of the shaft length.}$$

Also, angles of twist for *C*, *B*, *A* cross-sections we will calculate relative to reference *D*-section:

$$\varphi_C = \varphi_{DC} = \frac{M_x^{III}a}{GI_\rho} = \frac{-20\times10^3\times2\times32}{8\times10^{10}\times3.14\times0.131^4} = -1.73\times10^{-2} \text{ rad.}$$

$$\varphi_B = \varphi_{DB} = \varphi_{DC} + \varphi_{CB} = \varphi_{DC} + \frac{M_x^{II}b}{GI_\rho} = -1.73\times10^{-2} + \frac{-30\times10^3\times3\times32}{8\times10^{10}\times3.14\times0.131^4} =$$

$$= -1.73\times10^{-2} - 3.89\times10^{-2} = -5.62\times10^{-2} \text{ rad.}$$

$$\varphi_A = \varphi_{DA} = \varphi_{DB} + \varphi_{BA} = \varphi_{DB} + \frac{M_x^{I}c}{GI_\rho} = -5.62\times10^{-2} + \frac{40\times10^3\times4\times32}{8\times10^{10}\times3.14\times0.131^4} =$$

$$= -5.62\times10^{-2} + 6.92\times10^{-2} = +1.30\times10^{-2} \text{ rad.}$$

Corresponding graph is shown on Fig. 2.