LECTURE 13 Strength of a Bar in Pure Bending

Bending is a type of loading under which bending moments and also shear forces occur at cross sections of a rod. If the bending moment is the only force factor acting in the section while the shearing force is absent, bending is called **pure**.

In most cases, however, shearing forces occur as well as bending moments at cross sections of a rod. In this case bending is called **transverse**.

The examples of structural elements subjected to plane bending are shown in Figs 1–8.

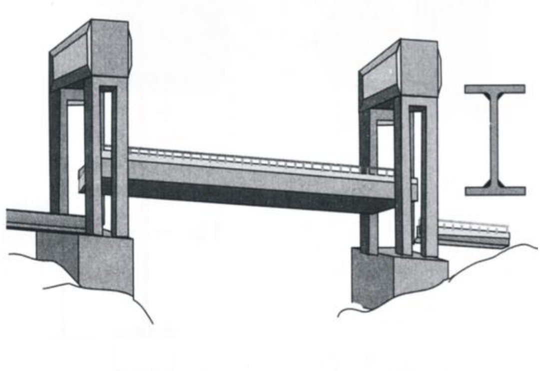

Fig. 1 A bridge with movable girder

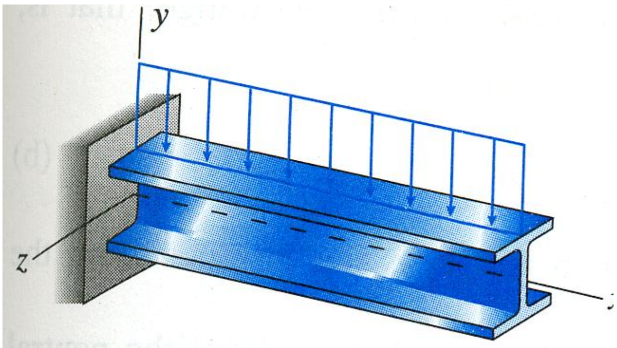

Fig. 2 Beam of wide-flanged shape loaded by distributed load

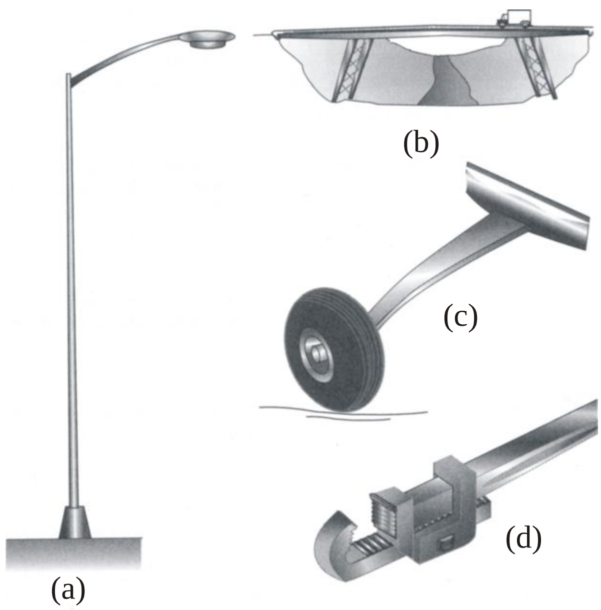

Fig. 3 Non-prismatic beams in bending: a) street lamp, b) bridge with tapered girders and pears, c) wheel strut of a small airplane, d) wrench handle

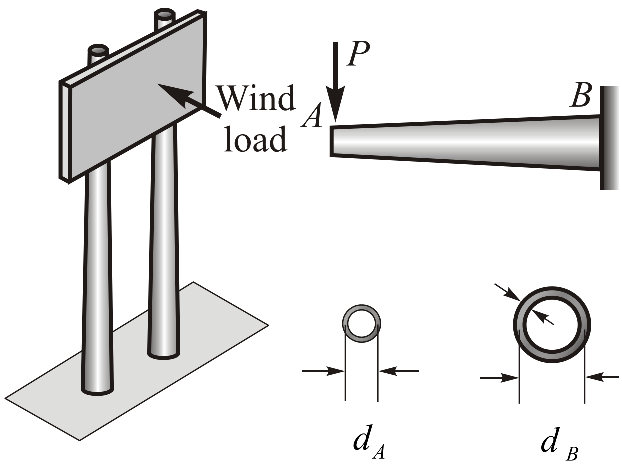

Fig. 4 A tall signboard supported by two vertical beams consisting of thin-walled, tapered circular tubes

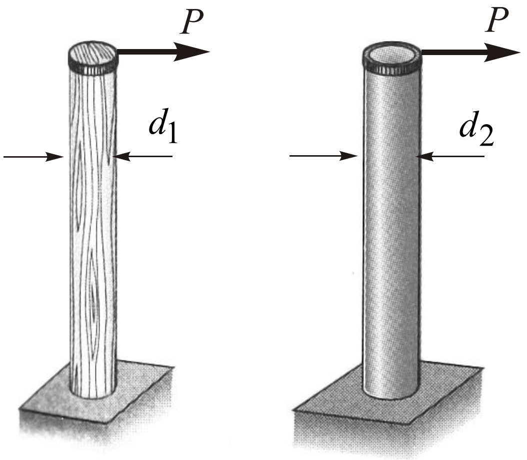

Fig. 5 Vertical solid wood and aluminum posts support a lateral load *P*

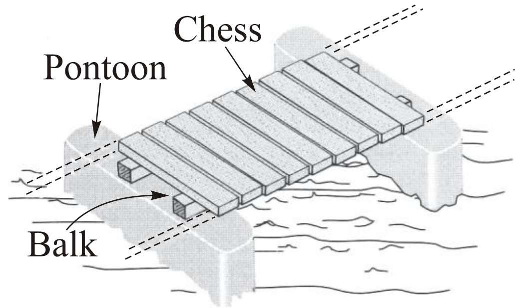

Fig. 6 A pontoon bridge consisting of two longitudinal wood beams (balks) that span between adjacent pontoons and support the transverse floor beams (cheeses)

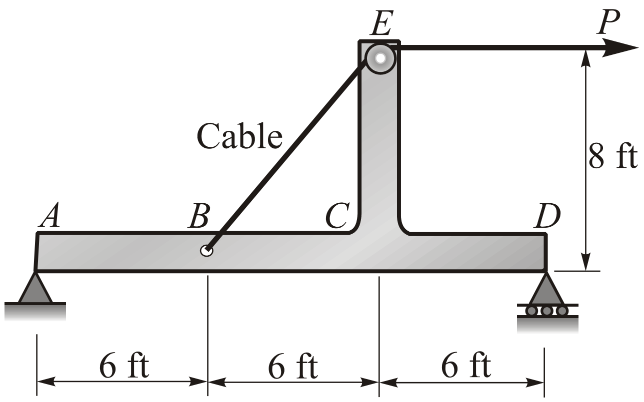

Fig. 7 Plane bending of a beam *ABCD* developed by the force *P* applied to the cable

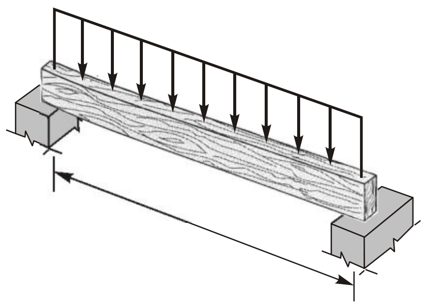

Fig. 8 Simply supported wood beam

Pure Bending

Let us consider an infinitesimally small element of a beam *dx* (see Fig. 9).

(1) The hypothesis of plane sections holds true for bending as well as for tension and torsion. Section *AB*, which was a plane before bending, has remained such after bending, but turned through a certain small angle $d\theta$.

(2) The plane *AB* continues to be perpendicular to the external surfaces of the beam. Therefore ***we have not the shearing stresses*** τ.

(3) The ***upper fibers of the beam became stretched***, i.e. increased in length from *ab* to *a'b'*, whereas the ***lower fibers became contracted*** from *ef* to *e'f'*. However the ***fibers don't press each other***. Therefore ***we haven't the normal stresses in perpendicular direction to the x axis***.

(4) There should be the line *cd* between the upper and lower fibers, which does not change its length on bending and is called the **neutral layer**.

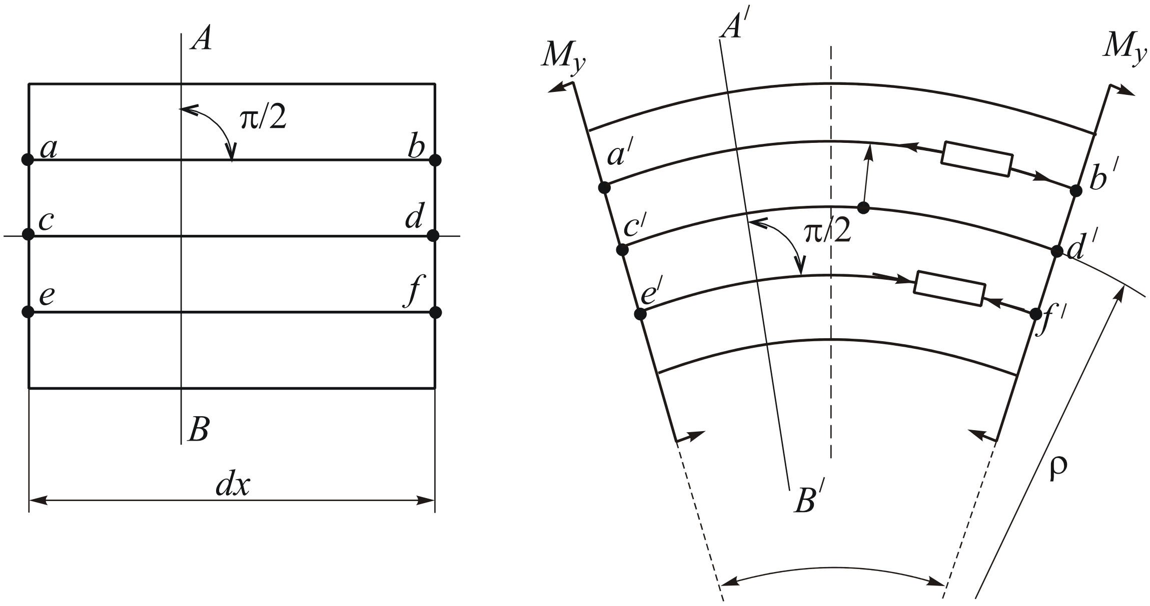

Fig. 9

Note: All the layers of the beam which are parallel to the neutral layer, are stretched or contracted and it is true the following relationship

$$\sigma_x = E\varepsilon_x . \quad (1)$$

In Fig. 9 the radius of the neutral layer is denoted by ρ.

1 Determination of Strains

Let us consider the deformation of particular fiber (layer) *ab* of the beam, which is at a distance *z* from the neutral layer *cd*.

The relative elongation of the layer *ab* is

$$\varepsilon_x = \frac{a'b' - ab}{ab}, \quad (2)$$

where $ab = dx = \rho d\theta = cd = c'd'$, $a'b' = (\rho + z)d\theta$,

whence

$$\varepsilon_x(z) = \frac{\rho d\theta + zd\theta - \rho d\theta}{\rho d\theta} = \frac{z}{\rho} = k_1 z. \qquad (3)$$

Note: the angle $d\theta$ is small, the arch $a'b'$ can be determined with a good accuracy as $a'b' = (\rho + z)d\theta$. Eq. 3 means, that the strains are distributed linearly between the layers.

2 Determination of the Neutral Axis Position

Due to tension-compression deformation of the fibers, the stresses can be found using Hooke's law:

$$\sigma_x(z) = \varepsilon_x(z)E, \qquad (4)$$

whence it follows after substitution:

$$\sigma_x(z) = E\frac{z}{\rho} = k_2 z. \qquad (5)$$

Thus ***in pure bending the stresses in the cross section vary according to a linear law*** with the proportionality factor k_2. The locus of points in the section which satisfies the condition $\sigma_x = 0$ is called the **neutral axis** of the section:

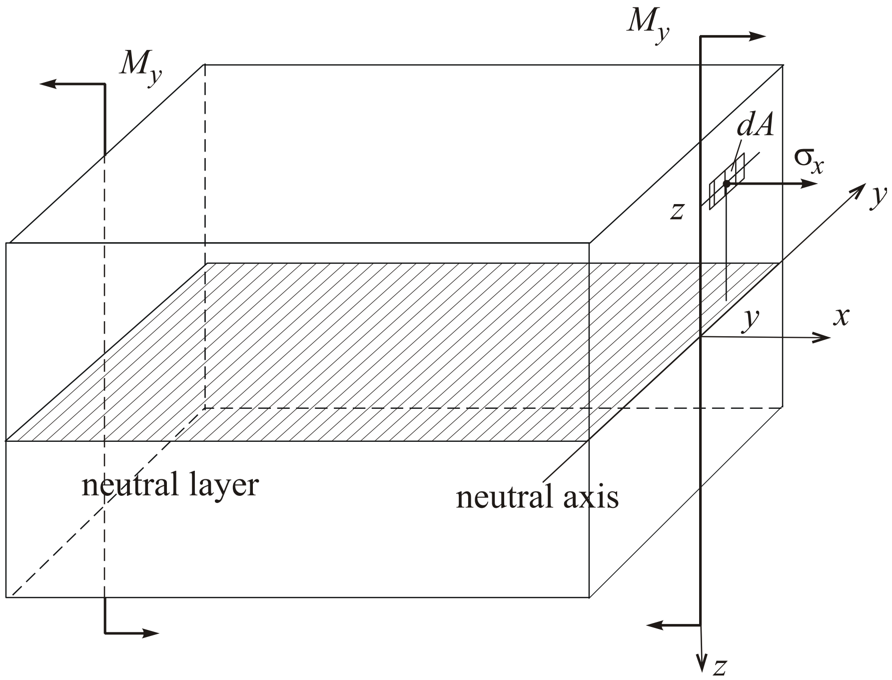

Fig. 10

Let us determine the position of the neutral axis. It may be recalled that the sum of the projections of all forces in a cross section onto the x axis is equal to zero, since ***there are no normal forces in the bending*** of the beam.

The elementary normal force acting on an elementary area

$$dN_x = \sigma_x dA = E\frac{z}{\rho}dA. \tag{6}$$

Summing over the entire area, we get:

$$N_x = \int_A \frac{Ez}{\rho}dA = 0. \tag{7}$$

Noting that the constant $\frac{E}{\rho}$ is other than zero, it follows:

$$\int_A zdA = 0. \tag{8}$$

This integral represents the static moment (first moment) of the section with respect to the neutral axis. Since the static moment is zero, the ***neutral axis passes through the centroid*** of a section (see Fig. 10).

Otherwise

$$M_z = \int_A \sigma_x ydA = 0, \tag{9}$$

$$M_z = \int_A \frac{E}{\rho}zydA = 0, \quad \frac{E}{\rho}\int_A zydA = 0, \quad \frac{E}{\rho} = const, \quad \int_A zydA = 0 = I_{yz} = 0.$$

Hence the ***y and z axes are the principal axes of the section***.

3 Determination of Stresses

For numerical determination of stresses, it is essential to find the radius of curvature ρ of the neutral layer of deflected beam.

Elementary moment relative to the neutral axis:

$$dM_y = \sigma_x zdA.$$

Summing the elementary moments over the cross-sectional area and substituting $\sigma_x = Ez/\rho$, we obtain:

$$M_y = \int_A \sigma_x z dA = \int_A \frac{Ez^2}{\rho} dA = \frac{E}{\rho} \int_A z^2 dA = \left\{ \int_A z^2 dA = I_y \right\} = \frac{E}{\rho} I_y ,$$

whence we find the **curvature** of deflected axis of beam:

$$\frac{1}{\rho} = \frac{M_y}{EI_y} . \tag{10}$$

Substituting the expression of curvature into the formula for σ_x, we finally get

$$\sigma_x = E\frac{z}{\rho} = Ez\frac{M_y}{EI_y} = \frac{M_y z}{I_y}; \quad \sigma_x = \frac{M_y z}{I_y} . \tag{11}$$

The ***maximum bending stress occurs at the points most remote from the neutral axis***:

$$\sigma_{x_{\max}} = \frac{M_y z_{\max}}{I_y} . \tag{12}$$

The quotient $I_y / z_{\max}$ is called the **section(al) modulus in bending** and is denoted by W_y:

$$W_y = \frac{I_y}{z_{\max}} , \text{[m}^3\text{]}. \tag{13}$$

Thus

$$\sigma_{\max} = \frac{M_y}{W_y} . \tag{14}$$

The diagram of bending stresses distribution along vertical z axis is shown in Fig. 11:

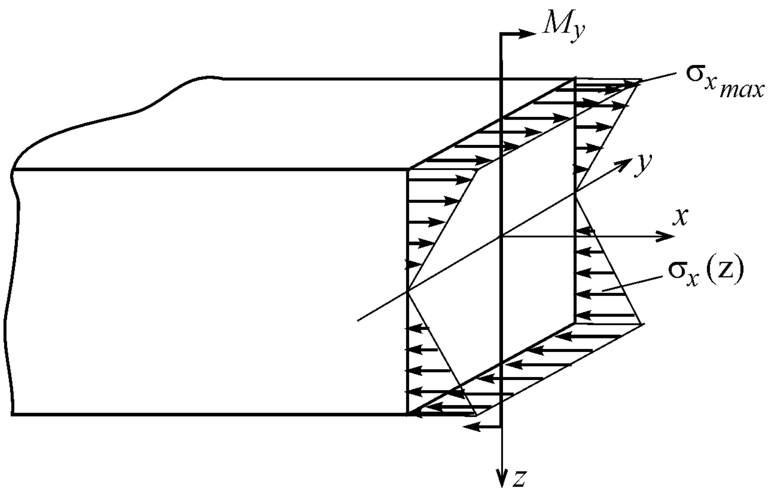

Fig. 11

Formula (13) is fundamental in the analysis of rods subjected to bending.

For a rod of rectangular section with sides *b* and *h*

$$I_y = \frac{bh^3}{12}, \quad z_{\max} = \frac{h}{2}, \quad W_y = \frac{bh^2}{6}.$$

For a circular section

$$I_y = \frac{\pi d^4}{64}, \quad z_{\max} = \frac{d}{2}, \quad W_y = \frac{\pi d^3}{32}.$$

4 Condition of Strength in Pure Bending

Condition of strength is really an inequality in which maximum working stresses and allowable stresses for structural element material are compared:

$$\sigma_{x_{\max}} = \frac{M_{y_{\max}}}{W_y} \le [\sigma]. \tag{15}$$

Considering the strength conditions, it is possible to solve three important engineering problems:

(1) ***problem of checking the strength.*** For specified loads and geometrical dimensions of a cross section the maximum stress in the section (called the **critical section**) is determined using the formula

$$\sigma_{\max} = \left(\frac{M_y}{W_y} \right)_{\max}$$

and compared with the allowable stress $[\sigma]$:

$$\sigma_{\max} \le [\sigma]; \qquad (16)$$

(2) ***design problem.*** For the specified loads and allowable stresses the cross-sectional area of a beam is determined by the formula:

$$W_y = \frac{M_{y\max}}{[\sigma]}; \qquad (17)$$

(3) ***problem of allowable load:***

$$[M_y] = W_y[\sigma], \qquad (18)$$

where $[M]$ is the allowable load determined for the critical section of a beam.

Example 1 Design problem in pure plane bending

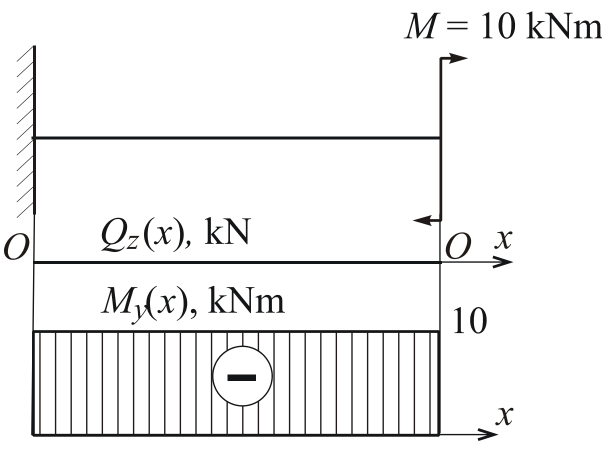

Fig. 12

For a cantilever beam select the following sections: a) circular section; b) rectangular section with $h/b = 2$; and c) I-section, using $[\sigma] = 100$ MPa.

Bending moments diagram shows the constant value of internal bending moment

$$|M_y| = 10 \text{ kNm}.$$

To solve the problem, it is necessary to calculate the section modulus W_y. Using condition of strength (14) we obtain

$$W_y \ge M_{y\max} / [\sigma] = 10 \times 10^3 / 100 \times 10^6 = 100 \times 10^{-6} \text{m}^3.$$

In the case (a) (round section) according to expression (14)

$$W_y^{\oslash} = \frac{\pi d^3}{32} = 100\times10^{-6} \rightarrow d = 10\times10^{-2}\,\text{m}, \quad A^{\oslash} = 80\times10^{-4}\,\text{m}^2,$$

In the case (b) (rectangle)

$$W_y^{\square} = \frac{bh^2}{6}; \quad h/b = 2; \quad \frac{b(2b)^2}{6} = 100\times10^{-6} \rightarrow b = 5.3\times10^{-2}\,\text{m};$$

$$h = 10.6\times10^{-2}\,\text{m}; \quad A^{\square} = 56.2\times10^{-4}\,\text{m}^2.$$

In the case (c)

$$W_y^{I} = 90.3\times10^{-6}\,\text{m}^3, \quad \text{I No14}, \quad A = 18.9\times10^{-4}\,\text{m}^2,$$

$$W_y^{I} = 118\times10^{-6}\,\text{m}^3, \quad \text{I No14}, \quad A = 21.5\times10^{-4}\,\text{m}^2,$$

Note, that $A^{\oslash} > A^{\square} > A^{\mathrm{I}}$ $(80 > 56.2 > 21.5)$

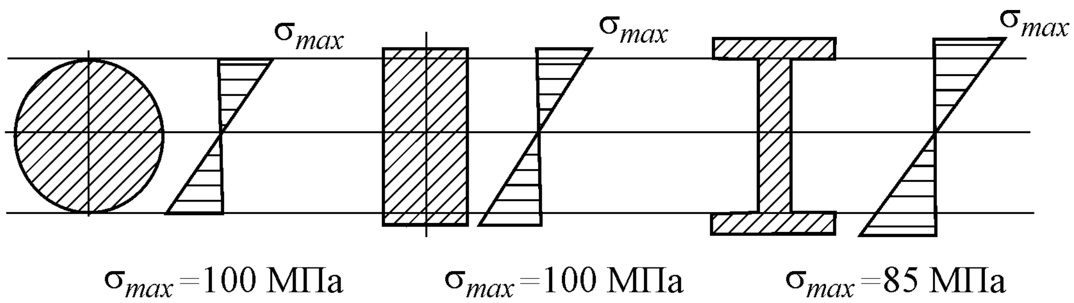

Fig. 13

Which of those sections is more advantageous from the viewpoint of strength-to-weight efficiency? Evidently, the I-section is the most efficient shape of cross section. For a beam to be efficient, most of the beam material should obviously be put as far as possible from the neutral axis. Remember, that ***for any section, the normal bending stress in a cross section of a beam is directly proportional to the distance from the neutral axis of the beam***.