LECTURE 15 Deflections of Beams

1 Introduction

When a beam with a straight longitudinal axis is loaded by lateral forces, the axis is deformed into a curve, called the **deflection curve** of the beam. In Lecture 13 we used the curvature of the deflection curve to determine the normal strains and stresses in a beam. In this lecture we will determine the equation of the deflection curve and find deflections at specified points along the axis of the beam. The calculation of deflections is an important part of structural analysis and design. For example, finding deflections is an essential ingredient in analysis of statically indeterminate structures. Deflections are also important in dynamic analysis.

2 Curvature of a Beam

When loads are applied to a beam, its longitudinal axis is deformed into a curve (see Fig. 1). As an example, consider a cantilever beam *AB* subjected to a load *F* at the free end (Fig. 1a):

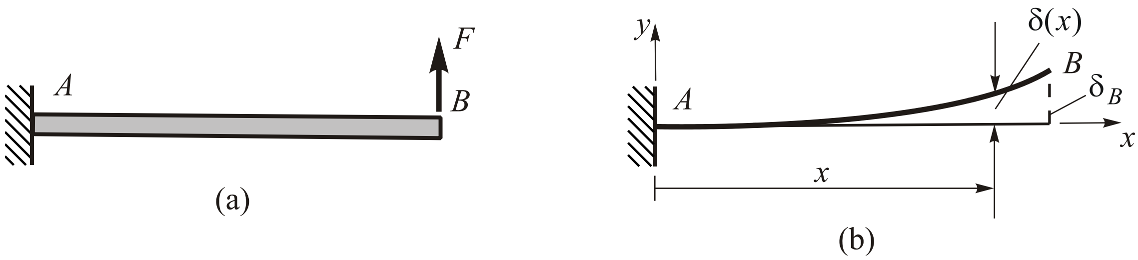

Fig. 1

Let us construct a system of coordinate axes (x, y) with the origin located at a suitable point on the longitudinal axis of the beam. We will place the origin at the fixed support. The positive x axis is directed to the right, and the positive y axis is directed upward.

The beam considered is assumed to be symmetric about the xy plane, which means that the y axis is an axis of symmetry of the cross section. In addition, all loads must act in the xy plane. As a consequence, the bending deflections occur in this same plane, known as the plane of bending.

The **deflection** of the beam at any point along its axis is the displacement of that point from its original position, measured in the *y* direction. We denote the deflection by the letter v. It is important to note that the resulting strains and stresses in the beam are directly related to the **curvature** of the deflection curve.

To illustrate the concept of curvature, consider again a cantilever beam subjected to a load *F* acting at the free end (Fig. 2). The deflection curve of this beam is shown in Fig. 2b. For purposes of analysis, we will identify two points m_1 and m_2 on the deflection curve. Point m_1 is selected at an arbitrary distance *x* from the *y* axis and point m_2 is located a small distance *ds* further along the curve. At each of this points we draw a line normal to the tangent to the deflection curve, that is, normal to the curve itself. This normals intersect at point O', which is the **center of curvature** of the deflection curve. Because most beams have very small deflections and nearly flat deflection curve, point O' is usually located much further from the beam that is indicated in the Fig. 2.

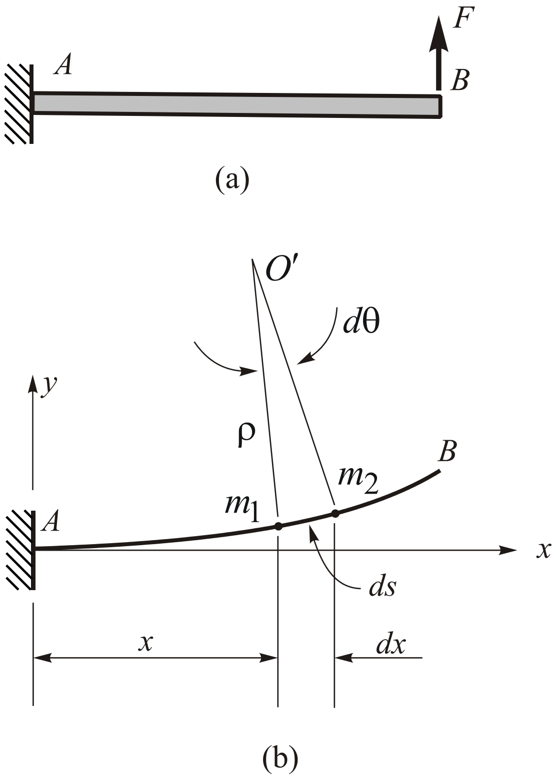

Fig. 2

The distance m_1O' from the curve to the centre of curvature is called the **radius of curvature** ρ, and the **curvature** *k* is defined as the reciprocal of the radius of curvature. Thus,

$$k = \frac{1}{\rho}. \qquad (1)$$

Curvature is a measure of how sharply a beam is bent. Both the curvature and the radius of curvature are functions of the distance *x*. It follows that the position O' of the center of curvature also depends upon the distance *x*.

We have proved in Lecture 13 that the curvature at a particular point on the axis of a beam depends on the bending moment at that point and on the properties of the beam itself: $1/\rho = M/EI$, where M is the bending moment and EI is the flexural rigidity of the beam. This equation is known as **moment-curvature equation**. Therefore, if the beam is prismatic and the material is homogeneous, the curvature will vary only with the bending moment. Consequently, a beam in pure bending will have constant curvature and the beam in transverse bending will have varying curvature.

The **sign convention for curvature** depends on the orientation of the coordinate axes. If the x axis is positive to the right and the y axis is positive upward, then the ***curvature is positive when the beam is bent concave upward (or convex downward)*** and the center of curvature is above the beam. Conversely, the curvature is negative when the beam is bent concave downward (or convex upward) and the center of curvature is below the beam. This sign convention is represented on the Fig. 3.

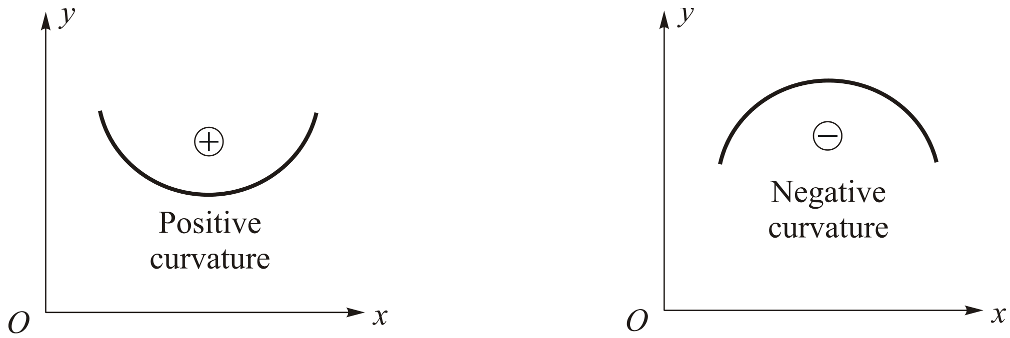

Fig. 3

2 Differential Equations of the Deflection Curve

Finding beam deflections are based on the differential equations of the deflection curve and their associated relationships. Consequently, we will begin by deriving the basic equation for the deflection curve of a beam.

Consider one more a cantilever beam with a concentrated load acting upward at the free end (Fig. 1). It was mentioned above that the deflection v is the displacement in the y-direction of any point on the axis of a beam (Fig. 1). Because the y axis is positive upward, the deflections are also positive upward. To obtain the equation of the deflection curve, we must express v as a function of *x*.

Let's consider the deflection curve on Fig. 1 in more detail. The deflection v at an arbitrary point m_1 on the deflection curve is shown in Fig. 4a. This point m_1 is located at distance *x* from the origin. A second point m_2, is located at distance $x+dx$ from the origin. The deflection at this second point is $v+dv$, where dv is the increment in deflection.

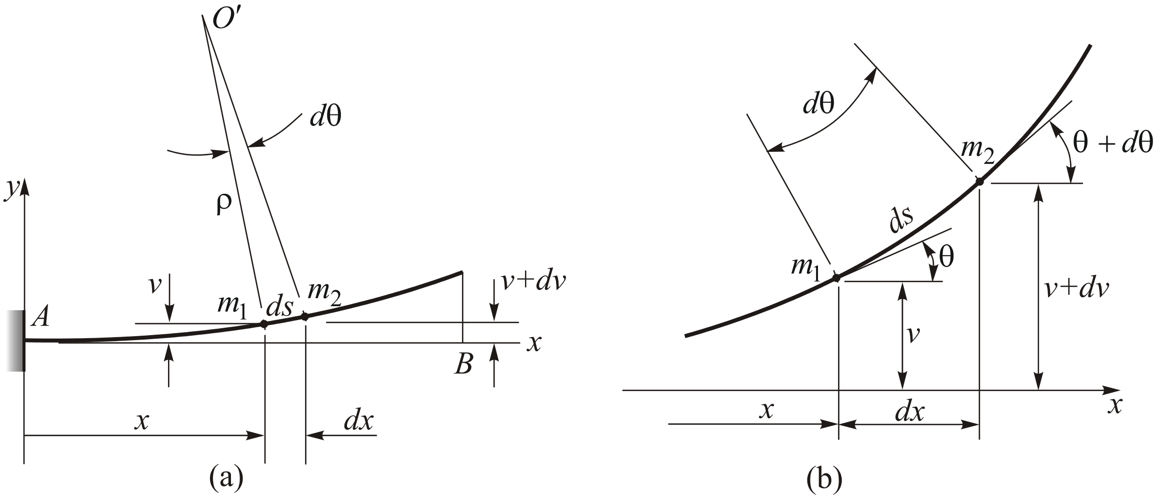

Fig. 4

When the beam is bent, there is not only a deflection at each point along the axis but also a rotation. The **angle of rotation** θ of the axis of the beam is the angle between the *x* axis and the tangent to the deflection curve, as shown for point m_1 in Fig. 4b. For our choice of axes (*x* positive to the right and *y* positive upward), the angle of rotation is positive when counterclockwise. (Other names for the angle of rotation are **angle of inclination** and **angle of slope**).

The angle of rotation at point m_2 is $\theta+d\theta$, where $d\theta$ is the increase in angle. It follows that if we construct lines normal to the tangents (see Fig. 4), the angle between these normals is $d\theta$. The point of intersection of these normals is the center of curvature O' and the distance from O' to the curve is the radius of curvature ρ. From Fig. 4a we see that

$$\rho d\theta = dS \, , \tag{2}$$

in which $d\theta$ is in radians and dS is the distance along the deflection curve between points m_1 and m_2. Therefore, the curvature k (equal to the reciprocal of the radius of curvature) is given by the equation:

$$k = \frac{1}{\rho} = \frac{d\theta}{dS} \, . \tag{3}$$

The sign convention for curvature is pictured in Fig. 3. ***Note that the curvature is positive when the angle of rotation increases as we move along the beam in the positive x direction***.

The slope of the deflection curve is the first derivative dv/dx of the expression for the deflection v. In geometrical terms, the slope is the increment dv in the deflection (as we go from point m_1 to point m_2) divided by the increment dx in the distance along the x axis. Since dv and dx are infinitesimally small, the slope dv/dx is equal to the tangent of the angle of rotation θ. Thus,

$$\frac{dv}{dx} = \tan\theta, \quad \theta = \arctan\frac{dv}{dx} \, . \tag{4}$$

In the similar manner, we also obtain the following relationships:

$$\cos\theta = \frac{dx}{ds}, \quad \sin\theta = \frac{dv}{ds} \, . \tag{5}$$

Note, that when the x and y axes have the directions shown in the Fig. 2, the slope dv/dx is positive when the tangent to the curve slopes upward to the right.

Equations (2) through (5) are based only on geometric considerations, and therefore they are valid for beams of any material. Furthermore, there are no restrictions on the magnitudes of the slopes and deflections. Most beams undergo very small deflections and angles of rotation, and so their deflection curves have extremely small curvatures. Therefore, we can simplify the analysis. For instance, since $\cos\theta \approx 1$ when θ is small, Eq. (5) gives

$$ds \approx dx \tag{6}$$

and Eq. (2) becomes

$$k = \frac{1}{\rho} = \frac{d\theta}{dx}. \tag{7}$$

Also, since $\tan\theta \approx \theta$ when θ is small, we can make the following approximation to Eq. (4):

$$\theta = \tan\theta = \frac{dv}{dx}. \tag{8}$$

Thus, if the rotations of a beam are small, we can assume that the angle of rotation θ and the slope dv/dx are equal.

Taking the derivative of θ with respect to *x* in Eq. (8), we get

$$\frac{d\theta}{dx} = \frac{d^2v}{dx^2}. \tag{9}$$

Combining these equations with Eq. (7), we obtain a relation between the curvature of the beam and its deflection:

$$k = \frac{1}{\rho} = \frac{d^2v}{dx^2}. \tag{10}$$

This equation is valid for a beam of any material and small rotations.

If the material of a beam .is linearly elastic and follows Hooke's law, the curvature is connected with bending moment:

$$k = \frac{1}{\rho} = \frac{M}{EI}. \tag{11}$$

Eq. (11) shows that a positive bending moment produces positive curvature and a negative bending moment produces negative curvature, as shown in Fig. 5:

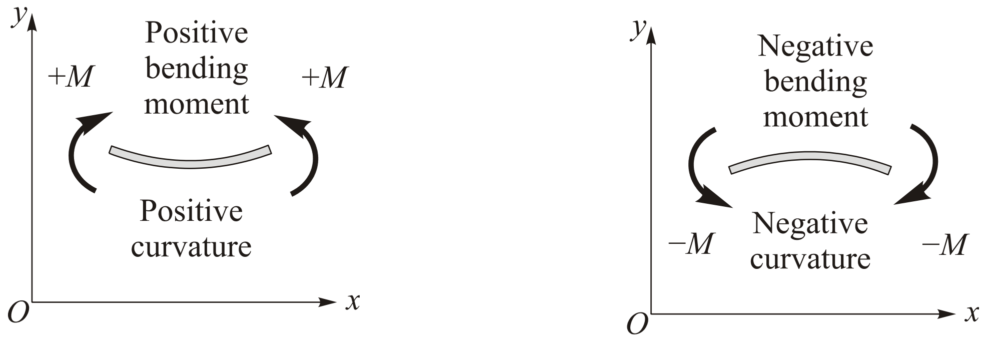

Fig. 5

Combining Eq. (10) with Eq. (11) yields the basic **differential equation of the deflection curve** of a beam:

$$\frac{d^2v}{dx^2} = \frac{M}{EI}. \tag{12}$$

This equation can be integrated in each particular case to find the deflection v, provided by bending moment M and flexural rigidity EI are known as functions of x.

As a reminder, the sign convention to be used with the proceeding equations are repeated here: (1) the x and y axes are positive to the right and upward, respectively; (2) the deflection v is positive upward; (3) the slope dv/dx and angle of rotation θ are positive when counterclockwise with respect to the positive x axis; (4) the curvature k is positive when the beam is bent concave upward; and (5) the bending moment M is positive when it produces compression in the upper part of the beam.