LECTURE 16   Deflections of Beams (Continued)

1   Calculation of Deflections by Integration of the Bending-Moment Equation

The first equation we will use is the bending-moment equation:

$$\frac{d^2v}{dx^2} = \frac{M}{EI}.$$ (1)

To simplify the writing of this equation, primes will be used to denote differentiation:

$$v' = \frac{dv}{dx}; \quad v'' = \frac{d^2v}{dx^2}.$$ (2)

Using this notation, we can express the differential equations for a prismatic beam in the following form:

$$EIv'' = M.$$ (3)

Since the Eq. (3) is of second order, two integrations are required. The first integration produces the slope $v' = dv/dx$, and the second produces the deflection v.

To begin the analysis we will write the equation (or equations) for the bending moments in the beam. In some cases a single bending-moment expression holds for the entire length of the beam. In other cases we must write separate bending-moment expressions for each portion of the beam between points where changes of the moment occur.

The general procedure for solving the differential equations is as follows. For each region of the beam, we substitute the expression for M into the differential equation and integrate to obtain the slope v'. Such integration produces one constant of integration. Next, we integrate each slope equation to obtain the corresponding deflection v. Again, each integration produces a new constant. Thus, there are two constants of integration for each region of the beam. These constants are evaluated from known conditions pertaining to the slopes and deflections. The conditions fall into three categories: (1) boundary conditions, (2) continuity conditions, and (3) symmetry conditions.

Boundary conditions pertain to the deflections and slopes at the supports of a beam. For example, at a simple support (either a pin or a roller) the deflection is zero, and at a fixed support both the deflection and the slope are zero. Each such boundary condition supplies one equation that can be used to evaluate the constants of integration.
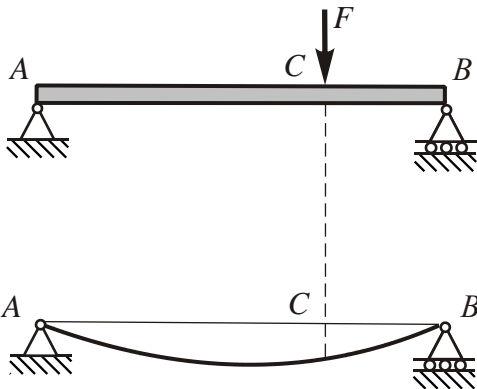
At point C: $(v)_{AC} = (v)_{CB}$

$(v')_{AC} = (v')_{CB}$

Fig. 1

Continuity conditions occur at points where the regions of integration meet, such as at point C in the beam of Fig. 6. The deflection curve of this beam is physically continues at point C, and therefore the deflection at point C as determined for the left-hand part of the beam must be equal to the deflection at point C as determined for the right-hand part. Similarly, the slopes found for each part of the beam must be equal at point C. Each of this continuity conditions supplies an equation for evaluating the constants of integration.
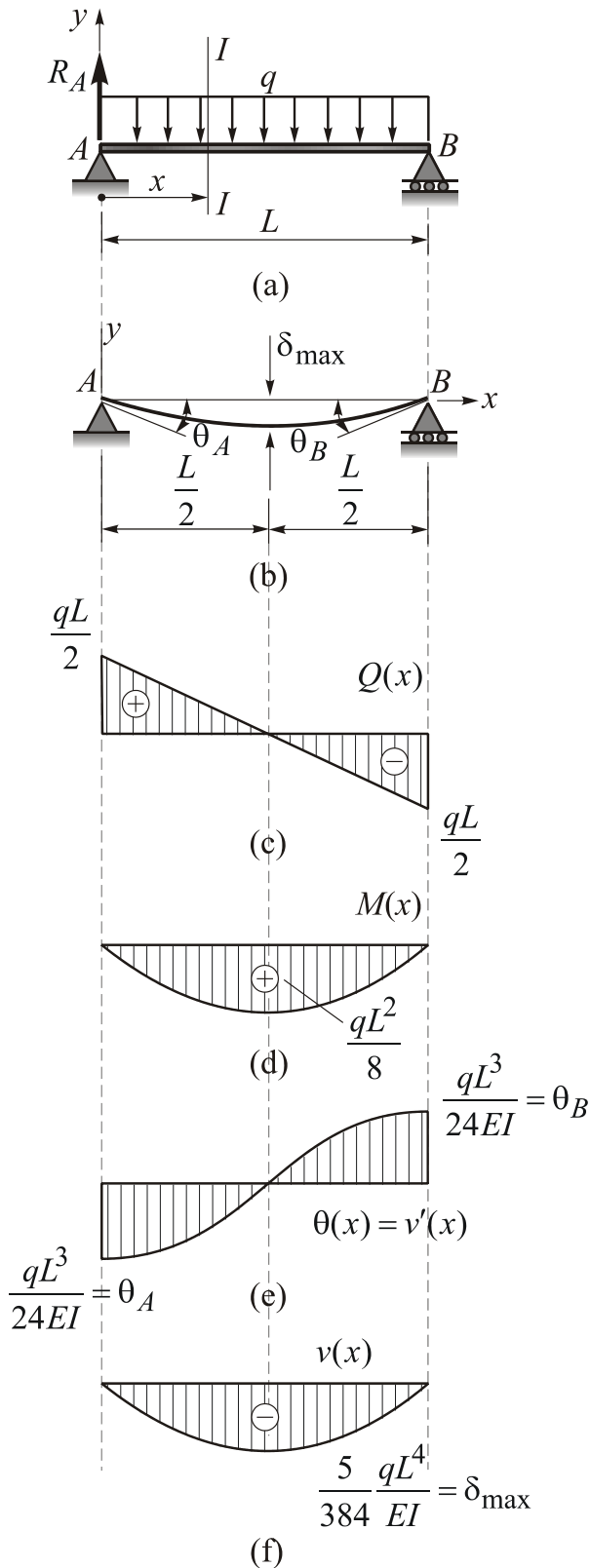
Symmetry conditions may also be available. For instance, if a simple beam supports a uniform load throughout its length, we know in advance that the slope of the deflection curve at the midpoint must be zero.

Each boundary, continuity, and symmetry condition leads to an equation containing one or more of the constants of integration. Since the number of independent condition always matches the number of constants of integration, we can always solve these equations for the constants.

Once the constants are evaluated, they can be substituted back into the expressions for slopes and deflections, thus yielding the final equations of the deflection curve. These equations can then be used to obtain the deflections and angles of rotation at particular points along the axis of the beam.

The proposed method for finding deflections is called the **method of successive integration**.

(a)

(b)

$\dfrac{qL}{2}$

$Q(x)$

(c)

$\dfrac{qL}{2}$

$M(x)$

(d)

$\dfrac{qL^2}{8}$

$\dfrac{qL^3}{24EI} = \theta_B$

$\theta(x) = v'(x)$

$\dfrac{qL^3}{24EI} = \theta_A$

(e)

$v(x)$

$\dfrac{5}{384}\dfrac{qL^4}{EI} = \delta_{\max}$

(f)

Example 1 Determine the equation of the deflection curve for a simple beam AB supporting a uniform load of intensity q. Also, determine the maximum deflection $\delta_{\max}$ at the midpoint of the beam and the angles of rotation at the supports. Assume, that the beam has length L and constant flexural rigidity EI.

Solution (1) *Bending moment in the beam.* Since the left reaction is $qL/2$, the equation for the bending moment is

$$M(x) = \frac{qL}{2}x - \frac{qx^2}{2}. \qquad (4)$$

(2) *Differential equation of the deflection curve.* By substituting the expression for the bending moment into the differential equation (3), we obtain

$$EIv''(x) = \frac{qLx}{2} - \frac{qx^2}{2}. \qquad (5)$$

This equation can be integrated to obtain the slopes and deflections of the beam.

(3) *Slope of the beam.* Multiplying both sides of the differential equation by dx and integrating, we get the following equation for the slope:

$$EIv'(x) = \frac{qLx^2}{4} - \frac{qx^3}{6} + C_1, \qquad (6)$$

in which C_1 is a constant of integration. From the symmetry of the beam, the slope of the deflection curve at midspan is equal to zero. Thus, we have the following symmetry conditions:

$$v' = 0 \quad \text{when} \quad x = L/2.$$

Applying this condition to Eq. (6) gives

$$0 = \frac{qL}{4}\left(\frac{L}{2}\right)^2 - \frac{q}{6}\left(\frac{L}{2}\right)^3 + C_1, \quad \text{or} \quad C_1 = -\frac{qL^3}{24}. \tag{7}$$

The equation for the slope of the beam then becomes

$$EIv'(x) = \frac{qLx^2}{4} - \frac{qx^3}{6} - \frac{qL^3}{24} \tag{8}$$

or

$$v'(x) = -\frac{q}{24EI}(L^3 - 6Lx^2 + 4x^3). \tag{9}$$

As expected, the slope is negative (i.e., clockwise) at the left-hand end of the beam ($x = 0$), positive at the right-hand end ($x = L$) and equal to zero at the midpoint ($x = L/2$).

(4) *Deflection of the beam.* The deflection is obtained by integrating the equation for the slope. Thus, upon multiplying both sides of Eq. (8) by *dx* and integrating, we obtain

$$EIv(x) = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24} + C_2. \tag{10}$$

The constant of integration C_2 may be evaluated from the condition that the deflection of the beam at the left-hand support is equal to zero; that is, $v = 0$ when $x = 0$, or $v(0) = 0$.

Applying this condition to a Eq. (10) yields $C_2 = 0$; hence the equation for the deflection curve is

$$EIv(x) = \frac{qLx^3}{12} - \frac{qx^4}{24} - \frac{qL^3x}{24} \tag{11}$$

or

$$v(x) = -\frac{qx}{24EI}(L^3 - 2Lx^2 + x^3). \tag{12}$$

This equation gives the deflection at any point along the axis of the beam. Note that the deflection is zero at both ends of the beam ($x = 0$ and $x = L$) and negative elsewhere (recall that downward deflections are negative).

(5) *Maximum deflection.* The maximum deflection $\delta_{\max}$ occurs at the midpoint of the span (see Fig. b) and is obtained by setting x equal to $L/2$ in Eq. (12). However, since $\delta_{\max}$ represents the magnitude of the maximum deflection, and since the deflection v is negative when downward, we must insert a minus sign in the equation, as follows:

$$\delta_{\max} = v\left(\frac{L}{2}\right) = -\frac{5qL^4}{384EI}. \tag{13}$$

(6) *Angles of rotation.* The maximum angles of rotation occur at the supports of the beam. At the left-hand end of the beam, the angle θ_A, which is clockwise angle (see Fig. b), is equal to the negative of the slope v'. Thus, by substituting $x = 0$ into Eq. (9), we find

$$\theta_A = v'(0) = -\frac{qL^3}{24EI}. \tag{14}$$

In the similar manner, we can obtain the angle of rotation θ_B at the right-hand end of the beam. Since θ_B is a counterclockwise angle, it is equal to the slope at the end:

$$\theta_B = v'(L) = +\frac{qL^3}{24EI}. \tag{15}$$

Because the beam and loading are symmetric about the midpoint, the angles of rotation at the angles are equal.

Example 2 Determine the equation of the deflection curve for a cantilever beam AB subjected to a uniform load of intensity q (see figure). Also, determine the angle of rotation θ_B and the deflection δ_B at the free end. The beam has length L and constant flexural rigidity EI.

Solution (1) *Bending moment in the beam.* The bending moment at distance x from the fixed support is obtained by the equation taking into account the vertical reaction at the support $R = qL$ and the moment reaction $qL^2/2$:

$$M(x) = -\frac{qL^2}{2} + qLx - \frac{qx^2}{2}. \tag{16}$$

(a)

(b)

(c)

(d)

(e)

(f)
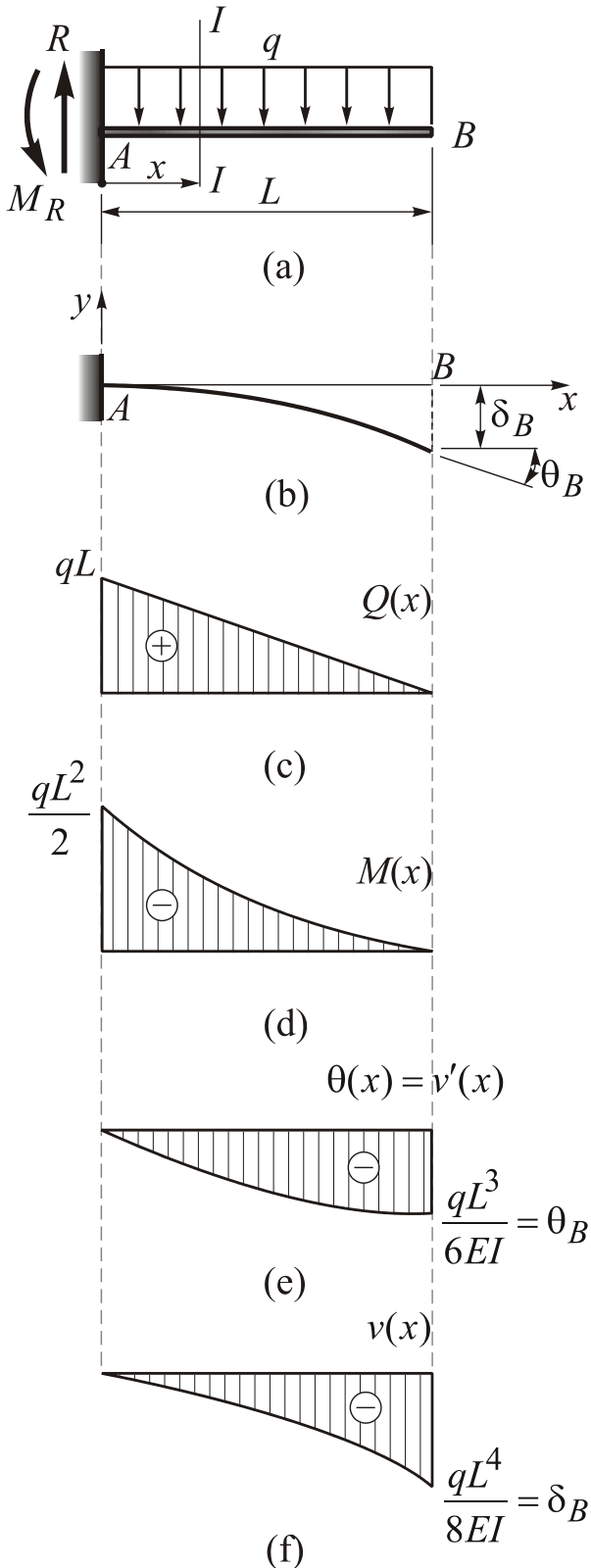
(1) *Differential equation of the deflection curve.* Substituting the preceding expression for the bending moment into the differential equation (3) we obtain

$$EIv''(x) = -\frac{qL^2}{2} + qLx - \frac{qx^2}{2}. \quad (17)$$

We now integrate both sides of this equation to obtain the slopes and deflections.

(2) *Slope of the beam.* The first integration of Eq. (17) gives the following equation for the slope:

$$EIv'(x) = -\frac{qL^2 x}{2} + \frac{qLx^2}{2} - \frac{qx^3}{6} + C_1 \quad (18)$$

The constant of integration C_1 can be found from the boundary condition that the slope of the beam is zero at the support: $v'(0) = 0$. When this condition is applied to Eq. (18) we get $C_1 = 0$. Therefore, Eq. (18) becomes

$$EIv'(x) = -\frac{qL^2 x}{2} + \frac{qLx^2}{2} - \frac{qx^3}{6} \quad (19)$$

and the slope is

$$v'(x) = -\frac{qx}{6EI}(3L^2 - 3Lx + x^2). \quad (20)$$

As expected, the slope is zero at the support ($x = 0$) and negative (i.e., clockwise throughout the length of the beam).

(3) *Deflection of the beam.* Integration of the slope equation (19) yields

$$EIv(x) = -\frac{qL^2 x^2}{4} + \frac{qLx^3}{6} - \frac{qx^4}{24} + C_2. \quad (21)$$

The constant C_2 is found from the boundary condition that the deflection of the beam is zero at the support: $v(0) = 0$. When this condition is applied to Eq. (21), we see immediately that $C_2 = 0$. Therefore, the equation for the deflection v is

$$v(x) = -\frac{qx^2}{24EI}(6L^2 - 4Lx + x^2).$$ (22)

As expected, the deflection v is zero at the support ($x = 0$) and negative (that is, downward) elsewhere.

(5) *Angle of rotation at the free end of the beam.* The clockwise angle of rotation θ_B at the end B of the beam (see figure) is equal to the negative of the slope at that point. Thus, using Eq. (20), we get

$$\theta_B = v'(L) = -\frac{qL^3}{6EI}.$$ (23)

This angle is the maximum angle of rotation for the beam.

(6) *Deflection at the free angle of the beam.* Since the deflection δ_B is downward (see figure) it is equal to the negative of the deflection obtained from Eq. (22):

$$\delta_B = v(L) = -\frac{qL^4}{8EI}.$$ (24)

This deflection is the maximum deflection of the beam.

Example 3 A simple beam AB supports a concentrated load P acting at distances a and b from the left-hand and right-hand supports, respectively (see figure). Determine the equations of the deflection curve, the angles of rotation θ_A and θ_B at the supports, the maximum deflection $\delta_{\max}$. Note that the beam has length L and constant flexural rigidity EI.

Solution (1) *Bending moments in the beam.* In this example the bending moments are expressed by two expressions, one for each part of the beam. Since the left reaction is Pb/L, we arrive at the following equations

$$M(x) = \frac{Pb}{L}x \quad (0 \le x \le a),$$ (25)

$$M(x) = \frac{Pbx}{L} - P(x-a) \quad (a \le x \le L).$$ (26)

(a)

(b)

(c)

(d)

(e)

(f)
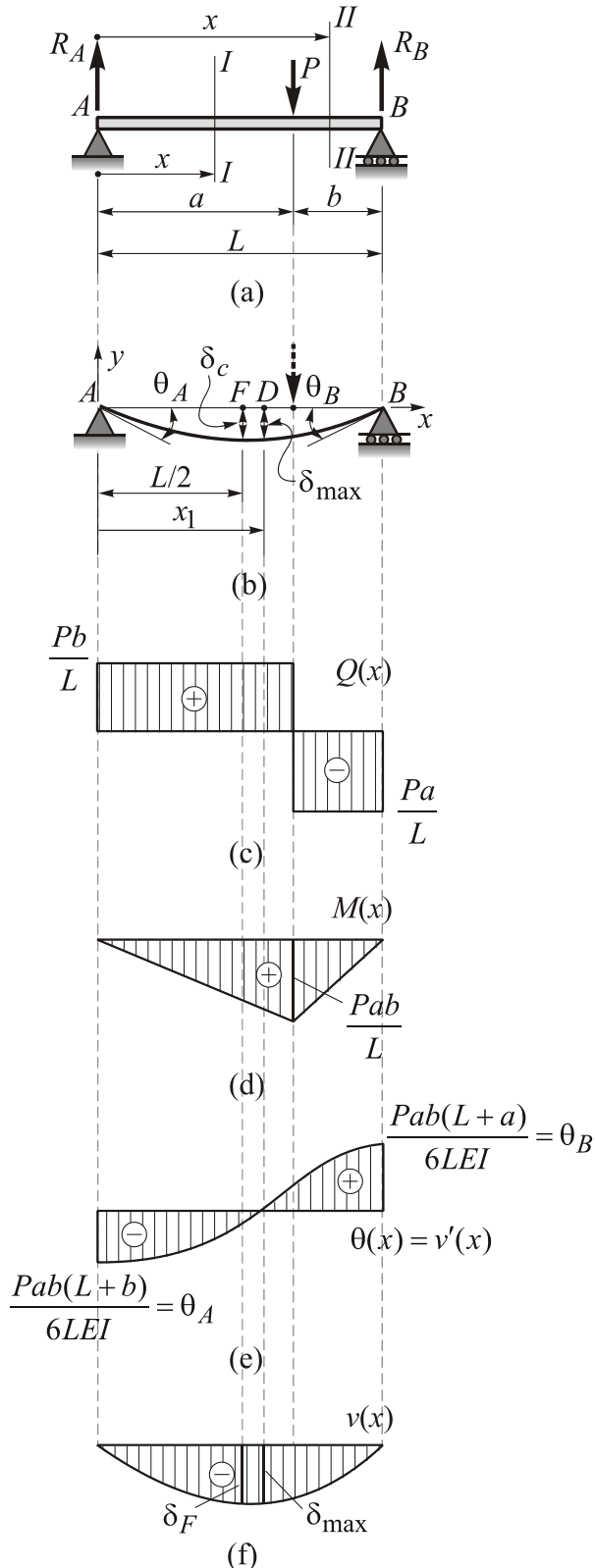
(1) *Slopes and deflections of the beam.* The first integrations of two differential equations yield the following expressions for the slopes

$$EIv'(x) = \frac{Pbx^2}{2L} + C_1 \quad (0 \le x \le a), \quad (27)$$

$$EIv'(x) = \frac{Pbx^2}{2L} - \frac{P(x-a)^2}{2} + C_2$$
$$(a \le x \le L), \quad (28)$$

in which C_1 and C_2 are constants of integration. A second pair of integrations gives the deflections:

$$EIv(x) = \frac{Pbx^3}{6L} + C_1 x + C_3$$
$$(0 \le x \le a), \quad (29)$$

$$EIv(x) = \frac{Pbx^3}{6L} - \frac{P(x-a)^3}{6} + C_2 x + C_4$$
$$(a \le x \le L). \quad (30)$$

These equations contain two additional constants of integration, making a total of four constants to be evaluated.

(2) *Constants of integration.* The four constants of integration can be find from the following four conditions:

(1) At $x = a$, the slopes v' for the two parts of the beam are the same;

(2) At $x = a$, the deflections v for the two parts of the beam are the same;

(3) At $x = 0$, the deflection v is zero;

(4) At $x = L$, the deflection v is zero.

The first two conditions are continuity conditions based upon the fact that the axis of the beam is a continuous curve. Conditions (3) and (4) are boundary conditions that must be satisfied at the supports.

Condition (1) means that the slopes determined from Eqs. (27) and (28) must be equal when $x = a$; therefore

$$\frac{Pba^2}{2L} + C_1 = \frac{Pba^2}{2L} + C_2 \quad \text{or} \quad C_1 = C_2. \tag{31}$$

Condition (2) means that the deflections found from Eqs. (29) and (30) must be equal when; therefore

$$\frac{Pba^3}{6L} + C_1 a + C_3 = \frac{Pba^3}{6L} + C_2 a + C_4. \tag{32}$$

This equation gives $C_3 = C_4$.

Next, we apply condition (3) to Eq. (29) and obtain $C_3 = 0$; therefore

$$C_3 = C_4 = 0. \tag{33}$$

Finally, we apply condition (4) to Eq. (30) and obtain

$$\frac{PbL^2}{6} - \frac{Pb^3}{6} + C_2 L = 0. \tag{34}$$

Therefore

$$C_1 = C_2 = -\frac{Pb\left(L^2 - b^2\right)}{6L}. \tag{35}$$

(4) *Equations of the deflection curve.* Let us substitute the constants of integrations (Eqs. (33) and (35)) and obtain the deflection equations for the two parts of the beam. The resulting equations, after a slight rearrangement, are

$$v(x) = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \quad (0 \le x \le a), \tag{36}$$

$$v(x) = -\frac{Pbx}{6LEI}(L^2 - b^2 - x^2) - \frac{P(x-a)^3}{6EI} \quad (a \le x \le L). \tag{37}$$

The first of these equations gives the deflection curve for the part of the beam to the left of the load P, and the second gives the deflection curve for the part of the beam to the right of the load.

      The slopes for the two parts of the beam can be found either by substituting the values of C_1 and C_2 into Eqs. (27) and (28) or by taking the first derivatives of the deflection (Eqs. (36) and (37)). The resulting equations are

$$v'(x) = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) \quad (0 \le x \le a), \tag{38}$$

$$v'(x) = -\frac{Pb}{6LEI}(L^2 - b^2 - 3x^2) - \frac{P(x-a)^2}{2EI} \quad (a \le x \le L). \tag{39}$$

The deflection and slope at any point along the axis of the beam can be calculated from Eqs. ((36), (37) and (38), (39).

(5) *Angles of rotation at the supports.* To obtain the angles of rotation θ_A and θ_B at the ends of the beam (see figure), we substitute $x = 0$ into Eq. (38) and $x = L$ into Eq. (39):

$$\theta_A = v'(0) = -\frac{Pb(L^2 - b^2)}{6LEI} = -\frac{Pab(L+b)}{6LEI}, \tag{40}$$

$$\theta_B = v'(L) = \frac{Pb(2L^2 - 3bL + b^2)}{6LEI} = \frac{Pab(L+a)}{6LEI}. \tag{41}$$

Note that the angle θ_A is clockwise (negative) and the angle θ_B is counterclockwise (positive) (see figure b).

      The angles of rotation are functions of the position of the load and reach their largest values when the load is located near the midpoint of the beam. In the case of the angle of rotation θ_A, the maximum value of the angle is

$$(\theta_A)_{max} = \frac{PL^2\sqrt{3}}{27EI} \tag{42}$$

and occurs when $b = L/\sqrt{3} = 0.577L$. This result is obtained by taking the derivative of θ_A (Eq. (40)) with respect to b and setting it equal to zero.

(6) *Maximum deflection of the beam.* The maximum deflection $\delta_{\max}$ occurs at point D (see figure b) where the deflection curve has a horizontal tangent. If the load is to the right of the midpoint, that is, if $a > b$, point D is in the part of the beam to the left of the load. We can locate this point by equating the slope v' from Eq. (49) to zero and solving for the distance x, which we now denote as x_1. In this manner we obtain the following formula for x_1:

$$x_1 = \sqrt{\frac{L^2 - b^2}{3}} \quad (a \geq b). \tag{43}$$

From this equation we see that as the load P moves from the middle of the beam ($b = L/2$) to the right-hand end ($b = 0$), the distance x_1 varies from $L/2$ to $L/\sqrt{3} = 0.577L$. Thus, the maximum deflection occurs at a point very close to the midpoint of the beam, and this point is always between the midpoint of the beam and the load.

The maximum deflection $\delta_{\max}$ is found by substituting x_1 (from Eq. (43)) into the deflection equation (Eq. (36)) and then inserting a minus sign:

$$\delta_{\max} = (v)_{x = x_1} = -\frac{Pb\left(L^2 - b^2\right)^{3/2}}{9\sqrt{3}LEI} \quad (a \geq b). \tag{44}$$

The minus sign is needed because the maximum deflection is downward (see figure b) whereas the deflection v is positive upward.

The maximum deflection of the beam depends on the position of the load P, that is, on the distance b. The maximum value of the maximum deflection (the "max-max" deflection) occurs when $b = L/2$ and the load is at the midpoint of the beam. This maximum deflection is equal to $PL^3/48EI$.