LECTURE 17 Strength Analysis in Combined Stresses. Basic Strength Theories

1 Application of Strength Theories

Strength theories are applied in strength analysis of structural elements under **combined stresses (non-uniaxial stress state)**.

2 Limiting State of Stress

It is usually assumed that, ***for a ductile material, the limiting state of stress is one which causes yielding, and for a brittle material, one which causes fracture***.

The limiting state of stress may be regarded as a characteristic of the material properties. ***When a structure is designed from the viewpoint of its strength, the state of stress at critical (dangerous) point of elastic solid under investigation is compared with the limiting state for the given material.*** Based on this comparison, the structure mechanical reliability is evaluated.

The first problem that arises in this method is the determination of the **limiting state of stress**. In the case of the uniaxial state of stress the problem is rather simple. The material is tested in tension. A characteristic point corresponding to the limiting stress of the given material is chosen on the **tension test diagram**. It is customary to take either the **yield point** $\sigma_{y,t}$ (for the ductile material) or the **ultimate tensile strength** $\sigma_{u,t}$ (for a brittle material) as the limiting stress state.

In the case of combined stresses it is technically difficult to test materials and there is endless number of possible types of state of stress in various structures. Hence it is necessary to develop a general method for assessing the **degree of danger** of combined stresses with a limited number of mechanical tests of the material. This is the subject of the theory of limiting states of stress or the so-called **theory of strength**.

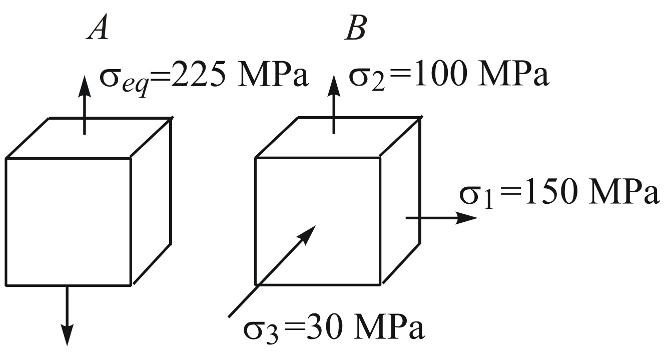
3 Equally Dangerous States of Stress

If the factors of safety are equal in two states of stress, the latter are referred to as equally dangerous. Thereafter the factor of safety in given state of stress will mean a

number indicating the ratio in which all the components of the state of stress should be increased simultaneously to make it limiting one.

Let's consider, for example, two states of stress *A* and *B* (see Fig. 1). If the yielding appears in element *A* of ductile material at $\sigma = 450\,\text{MPa}$ applied, this state of stress is called as limiting.

If, for example, the yielding appears in element *B* of ductile material at the combination of applied stresses $\sigma_1 = 300\,\text{MPa}$, $\sigma_2 = 200\,\text{MPa}$, $\sigma_3 = -60\,\text{MPa}$, this state of combined stresses will be limiting too.



State *A* is as dangerous as state *B* because both states have the same factor of safety equals to 2.0

Fig. 1

If, for example, it is necessary to provide the factor of safety $n = 2$, in both cases, two states of stress:

a) uniaxial with the stress $\sigma = 225\,\text{MPa}$ and

b) combined with the combination of stresses $\sigma_1 = 150\,\text{MPa}$, $\sigma_2 = 100\,\text{MPa}$ and $\sigma_3 = -30\,\text{MPa}$ are **equally dangerous**.

4 Equivalent Stress

The equivalent stress σ_{eq} is a stress that must be produced in a tensile test specimen to make its state of stress as dangerous as the given one. If the value of σ_{eq} is found, i.e. expressed in terms of σ_1, σ_2 and σ_3 the problem of the degree of danger of the combined stresses may be considered as solved, because the factor of safety in tension (state *A* in Fig.1) is given, as usual, by $n = \dfrac{\sigma_{y,t}}{\sigma_{eq}}$.

Since the factor of safety should have the same value in the case of the combined state of stress *B*, the problem of structural design based on the principle of the maximum stresses estimation and limitation under combined stress state is simplified to the familiar design under simple tension.

The only problem is how to express σ_{eq} in terms σ_1, σ_2 and σ_3. To solve it, we shall consider some of the theories (hypotheses) of strength.

5 Basic Theories of Strength

I The Maximum Normal Stress Theory

State A will be as dangerous as state B (Fig. 1) if the maximum normal stress σ_1 is equal to the limiting stress $\sigma_{\mathbf{lim}}$, which is determined from the tension test:

$$\sigma_{\max} = \sigma_1 = \sigma_{\lim} \text{ – equation of limiting state of stress.}$$

If the factor of safety is equal to *n*, then

$$\sigma_1 = \frac{\sigma_{\lim}}{n} = [\sigma]$$

and we have the **condition of strength** with the factor of safety *n*:

$$\sigma_{eq}^{I} = \sigma_1 \le [\sigma], \text{ where } [\sigma] = \frac{\sigma_{\lim}}{n} \text{ – allowable stress.} \qquad (1)$$

Practical verification did not support this hypothesis, excluding small range of brittle materials.

II The Maximum Linear Strain Theory

State A will be as dangerous as state B with the factor of safety $n = 1$ if the maximum linear strain is equal to the limiting one, which is determined from the tension test:

$$\varepsilon_{\max} = \varepsilon_1 = \varepsilon_{\lim} \text{ – equation of limiting stress state,}$$

or, in accordance with the generalized Hook's law,

$$\frac{1}{E}[\sigma_1 - \mu(\sigma_2 + \sigma_3)] = \frac{1}{E}\sigma_{\lim}.$$

Condition of strength with the factor of safety *n* is:

$$\sigma_{eq}^{II} = [\sigma_1 - \mu(\sigma_2 + \sigma_3)] = \frac{\sigma_{\lim}}{n} \le [\sigma]. \qquad (2)$$

This theory was widely applicable in XIX century, but a detailed verification revealed a number of serious defects.

III The Maximum Shearing Stress Theory

State A will be as dangerous as state B with the factor of safety $n = 1$ if the maximum shearing stresses attain a limiting value, which is determined from the tension test:

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \tau_{\lim} \text{ – equation of limiting stress state,}$$

where $\tau_{\lim} = \sigma_{\lim}/2$. It means that

$$\frac{1}{2}(\sigma_1 - \sigma_3) = \frac{\sigma_{\lim}}{2}.$$

Condition of strength with the factor of safety *n* is:

$$\sigma_{eq}^{III} = (\sigma_1 - \sigma_3) \le [\sigma]. \qquad (3)$$

Experimental verification of this expression under various conditions of stress has shown that, in general, it leads to satisfactory results only for ductile materials.

IV The Distortion Energy Theory

State A will be as dangerous as state B with the factor of safety $n = 1$, if the distortion energy attains a limiting value, which is determined from the tension test diagram.

(a) **Determination of Potential Energy of Strain (Strain Energy)**

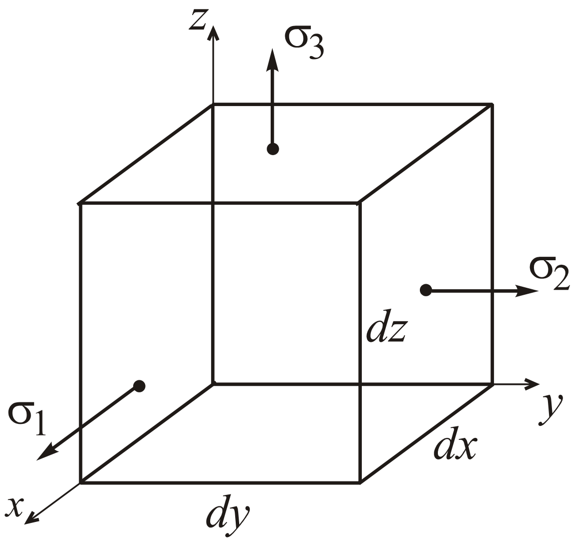


Fig. 2

The **potential energy** stored in an elementary volume of elastic solid in its deformation is determined by the total **work** done by the forces distributed over the surfaces of this volume. The normal force $\sigma_1 dydz$ does work on the displacement $\Delta dx = \varepsilon_1 dx$. This **work of elastic force** is given by formula

$$\frac{1}{2}\Delta dx(\sigma_1 dydz)\left\{\frac{\Delta dx}{dx} = \varepsilon_1\right\} = \frac{1}{2}\sigma_1\varepsilon_1 dxdydz,$$

where ε_1 is the strain along the *x* axis.

Similar expressions are obtained for the work done by the other normal components: $\frac{1}{2}\sigma_2 dzdx\varepsilon_2 dy$; $\frac{1}{2}\sigma_3 dxdy\varepsilon_3 dz$.

Then total **strain energy** stored in an elementary volume $dxdydz$ is

$$dU = \frac{1}{2}dxdydz\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right).$$

If the energy is estimated per unit volume, we get so called **strain energy density**

$$u_0 = \frac{dU}{dV} = \frac{1}{2}\left(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3\right).$$

Let's express strains in terms of stresses by generalized Hook's law:

$$\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right],$$

$$\varepsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\left(\sigma_1 + \sigma_3\right)\right],$$

$$\varepsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\left(\sigma_1 + \sigma_2\right)\right].$$

Then

$$u_0 = \left\{\sigma_1\frac{1}{E}\left[\sigma_1 - \mu\left(\sigma_2 + \sigma_3\right)\right] + \sigma_2\frac{1}{2}\left[\sigma_2 - \mu\left(\sigma_3 + \sigma_1\right)\right] + \sigma_3\frac{1}{E}\left[\sigma_3 - \mu\left(\sigma_1 + \sigma_2\right)\right]\right\},$$

$$u_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right)\right]. \qquad (4)$$

Strain energy describes the possibility of elastic solid to change its shape and also volume in deformation. We shall derive the expression for the energy of changing the shape, so-called **distortion energy**, and the **energy of changing the volume (dilatation)**. For this purpose, the principal stresses are each represented as the sum of two quantities – **spherical** and **complementary**:

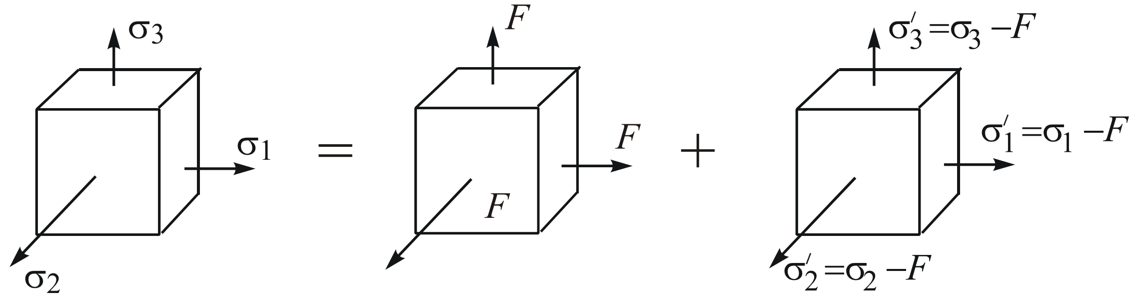


Fig. 3

$$\begin{cases} \sigma_1 = F + \sigma_1' \\ \sigma_2 = F + \sigma_2' \\ \sigma_3 = F + \sigma_3' \end{cases} \qquad (5)$$

The magnitude of F is chosen so that ***there will be no change of volume in the complementary state of stress***, i.e.

$$e_V = \frac{dV}{V_0} = \frac{1-2\mu}{E}\left(\sigma_1' + \sigma_2' + \sigma_3'\right) = 0. \qquad (6)$$

By adding expression (5), we obtain

$$F = \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right). \qquad (7)$$

Thus, the internal energy is divided into two parts corresponding to the two states of stress

$$U_0 = U_{0V} + U_{0S},$$

where U_{0V} is the energy of changing the volume, U_{0S} is the energy of changing the shape.

By substituting the quantity F from (7) in expression (4) we find

$$U_{0V} = \frac{1-2\mu}{6E}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2.$$

The distortion energy is found by subtracting U_{0V} from U_0. After simplifications we obtain

$$U_{0S} = \frac{1+\mu}{6E}\left[\left(\sigma_1 - \sigma_2\right)^2 + \left(\sigma_2 - \sigma_3\right)^2 + \left(\sigma_3 - \sigma_1\right)^2\right]. \qquad (8)$$

(b) **Condition of Strength in Accordance with Distortion Energy Theory**

According to the distortion energy theory, combined state of stress will be as dangerous as uniaxial with the factor of safety $n=1$, if $\frac{1+\mu}{6E}\left[\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2\right]=\frac{1+\mu}{6E}2\sigma_{\lim}^2$ – **equation of limiting stress state**.

If the factor of safety is equal to *n*, we get

$$\frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2}=\frac{\sigma_{\lim}}{n}$$

and we have the condition of strength with the factor of safety *n* as follows:

$$\sigma_{eq}^{IV}\frac{1}{\sqrt{2}}\sqrt{\left(\sigma_1-\sigma_2\right)^2+\left(\sigma_2-\sigma_3\right)^2+\left(\sigma_3-\sigma_1\right)^2}\le[\sigma]. \qquad (9)$$

The result obtained is numerically close to that given by the maximum shearing stress theory.