LECTURE 18   Combined Loading of a Bar System (part 1)

1   General Case of Combined Stresses

Let us consider the general case of the rod loading when normal and shearing forces, bending and twisting moments may occur simultaneously in cross sections of a rod ($N_x \neq 0$, $Q_y \neq 0$, $Q_z \neq 0$, $M_x \neq 0$, $M_y \neq 0$, $M_z \neq 0$).
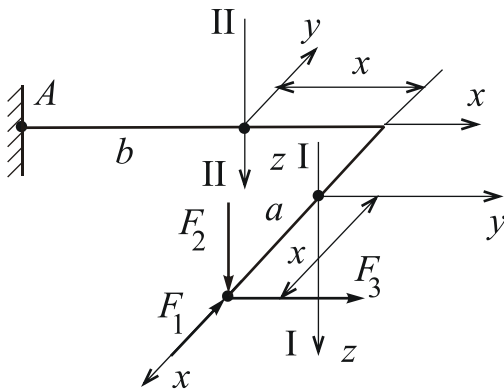
Fig. 1

The determination of critical sections and points is preceded by the analysis of internal forces induced in the rod.

This analysis is made, by using the method of sections and is completed by the construction of bending and twisting moment diagrams together with normal and shearing force diagram.

In all cases, diagrams of internal forces distribution are constructed on the axial line of a rod. The magnitude of the force is laid off along a normal to the axial line. For a three-dimensional rod, the axial line is usually drawn in perspective and two bending moment diagrams are given in the corresponding planes of bending.
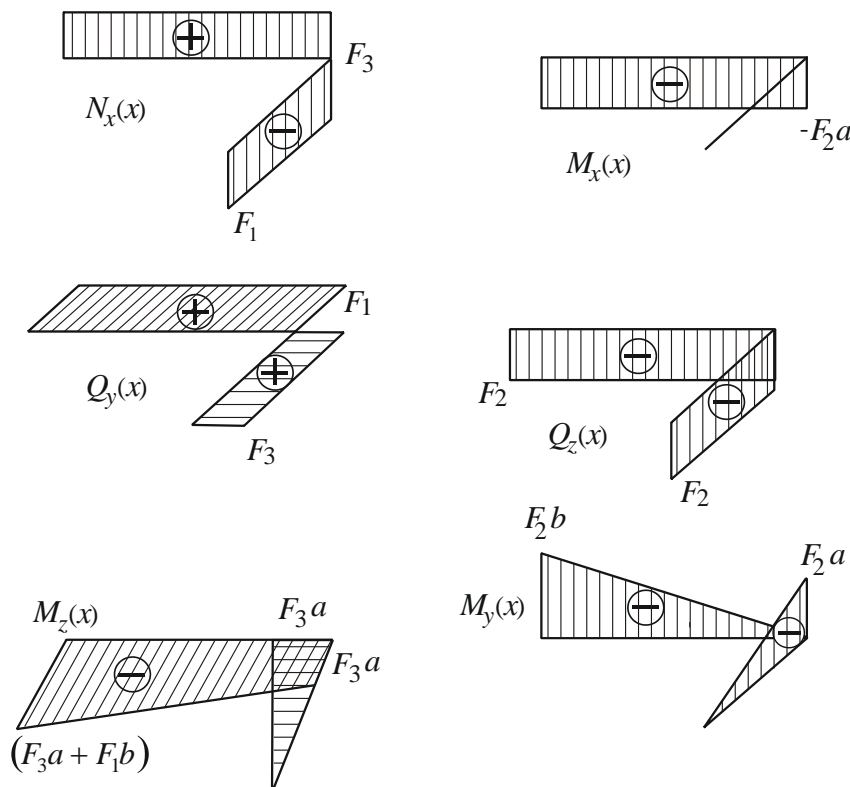
Fig. 2

$$N_x^I(x) = -F_1; \quad Q_y^I(x) = +F_3; \quad Q_z^I(x) = F_1;$$

$$N_x^{II}(x) = +F_3; \quad Q_z^{II}(x) = -F_2; \quad Q_y^{II}(x) = -F_1;$$

$$M_x^I(x) = 0; \quad M_y^I(x) = -F_2 x\,|_{x=0} = 0\,|_{x=a} = -F_2 a;$$

$$M_x^{II}(x) = -F_2 a; \quad M_y^{II}(x) = -F_2 x\,|_{x=0} = 0\,|_{x=b} = -F_2 b;$$

$$M_z^I(x) = -F_3 x\,|_{x=0} = 0\,|_{x=a} = -Fa;$$

$$M_z^{II}(x) = -F_3 a - F_1 x\,|_{x=0} = -F_3 a\,|_{x=b} = -F_3 a - F_1 b.$$

In order to check the strength of the beam we must find in the diagrams the critical (dangerous) section of the rod, i.e. the section where the maximum internal forces developing the dangerous state of stress are applied. This section in our case coincides with the built-in (rigidly fixed) support.

Because of the small magnitude of $\tau_{\max}(Q_z)$ and $\tau_{\max}(Q_y)$ in solid cross-sections the analysis of rods subjected to transverse bending is made only on the basis of normal stresses as in pure bending. That is why shearing stresses $\tau_{\max}(Q_z)$, $\tau_{\max}(Q_y)$ are not taken into consideration.

2　Bending, Torsion and Tension of a Rectangular Rod

As it is seen from the internal force diagrams, critical section of second portion coincides with rigidly fixed support. It means that we have the following combination of internal forces:
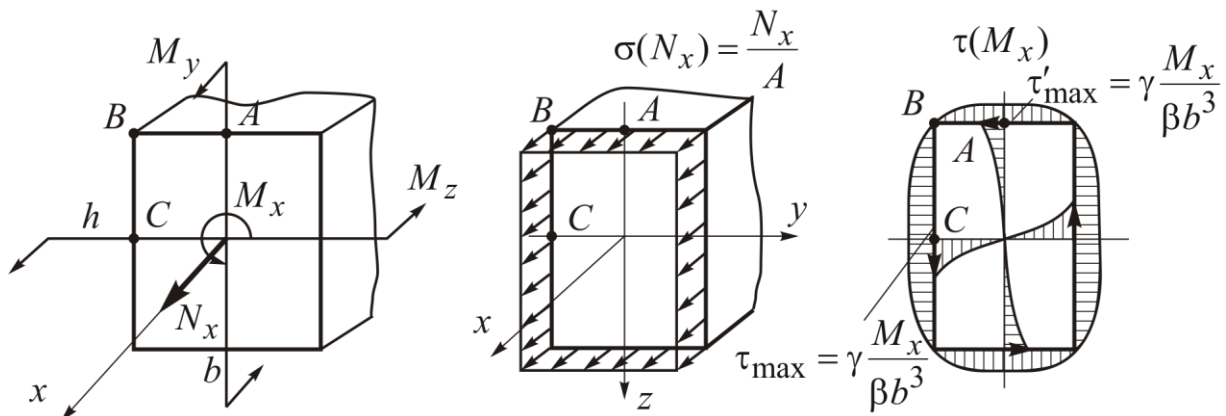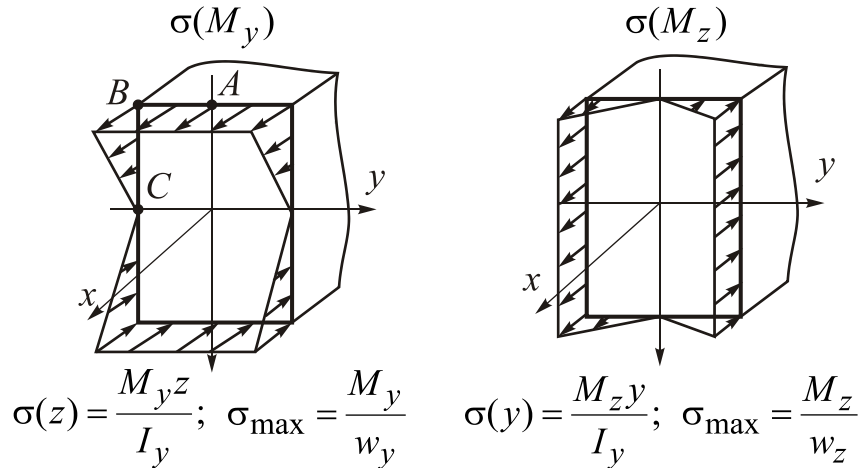

Fig. 3

$$\sigma(M_y)$$ $$\sigma(M_z)$$

$$\sigma(z) = \frac{M_y z}{I_y}; \quad \sigma_{\max} = \frac{M_y}{w_y} \qquad \sigma(y) = \frac{M_z y}{I_y}; \quad \sigma_{\max} = \frac{M_z}{w_z}$$

Fig. 4

As may be seen from Fig. 3 the maximum normal stresses should appear in the corner points of the section. It means that *first potentially dangerous point is located at the point B*.

$$\sigma_{x_{\max}} = \sigma_B = \frac{N_x}{A} + \frac{M_y}{W_y} + \frac{M_z}{W_z}. \tag{1}$$

Point B

Fig.5

But it should be noted here that the shearing stress at the point B is equal to zero:

$$\tau_{p.B} = 0.$$
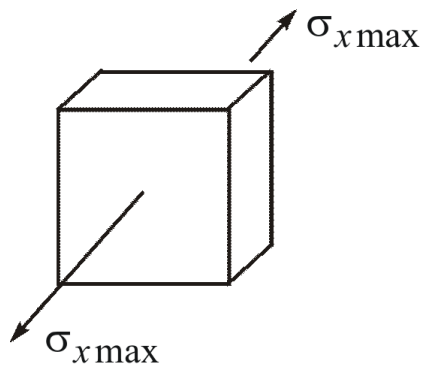
Conclusion: *there is uniaxial stress(ed) state at the point B* (see Fig. 5). The condition of strength is simple contrast of maximum acting stress with allowable one:

$$\sigma_{x_{\max}} = \frac{N_x}{A} + \frac{M_y}{W_y} + \frac{M_z}{W_z} \leq [\sigma]. \tag{2}$$

Other potentially critical points are the points A and C:
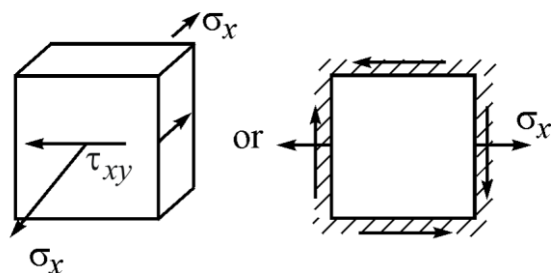
Point A

Fig.6

$$\sigma_A = \frac{N_x}{A} + \frac{M_y}{W_y};$$

$$\tau_A = \tau'_{\max} = \gamma \frac{M_x}{\beta b^3}.$$

Let us determine the principal stresses:

$$\sigma_{1,2(3)} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} \ .$$

$$\sigma_1 = \frac{\sigma}{2} + \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}, \quad \sigma_2 = 0,$$

$$\sigma_3 = \frac{\sigma}{2} - \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2} \ .$$

It should be observed that ***there is biaxial stress(ed) state at the point*** A. According to the maximum shearing stress theory of strength

$$\sigma^{III}_{eq_{p.A}} = (\sigma_1 - \sigma_3) \le [\sigma] \tag{3}$$

or

$$\sigma^{III}_{eq_{p.A}} = \sqrt{\sigma^2 + 4\tau^2} \le [\sigma], \quad \sigma^{III}_{eq_{p.A}} = \sqrt{\left(\frac{N_x}{A} + \frac{M_y}{W_y}\right)^2 + 4\left(\gamma\frac{M_x}{\beta b^3}\right)^2} \le [\sigma]. \tag{4}$$

According to the distortion energy theory of strength

$$\sigma^{IV}_{eq_{p.A}} = \sqrt{\sigma^2 + 3\tau^2} = \sqrt{\left(\frac{N_x}{A} + \frac{M_y}{W_y}\right)^2 + 3\left(\gamma\frac{M_x}{\beta b^3}\right)^2} \le [\sigma]. \tag{5}$$

Point C
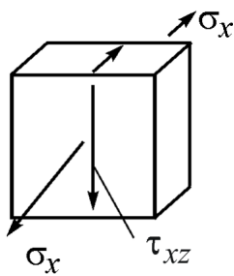

Fig.7

$$\sigma_c = \frac{N_x}{A} + \frac{M_z}{W_z};$$

$$\tau_c = \tau_{\max} = \frac{M_x}{\beta b^3}.$$

By analogy with the point A

$$\sigma^{III}_{eq_{p.C}} = \sqrt{\left(\frac{N_x}{A} + \frac{M_z}{W_z}\right)^2 + 4\left(\frac{M_x}{\beta b^3}\right)^2} \le [\sigma], \tag{6}$$

$$\sigma^{IV}_{eq_{p.C}} = \sqrt{\left(\frac{N_x}{A} + \frac{M_z}{W_z}\right)^2 + 3\left(\frac{M_x}{\beta b^3}\right)^2} \le [\sigma]. \tag{7}$$

It is usual to use formula (2) for determining the dimensions of rectangular sections with following checking the strength using formulae (4, 6 or 5, 7).

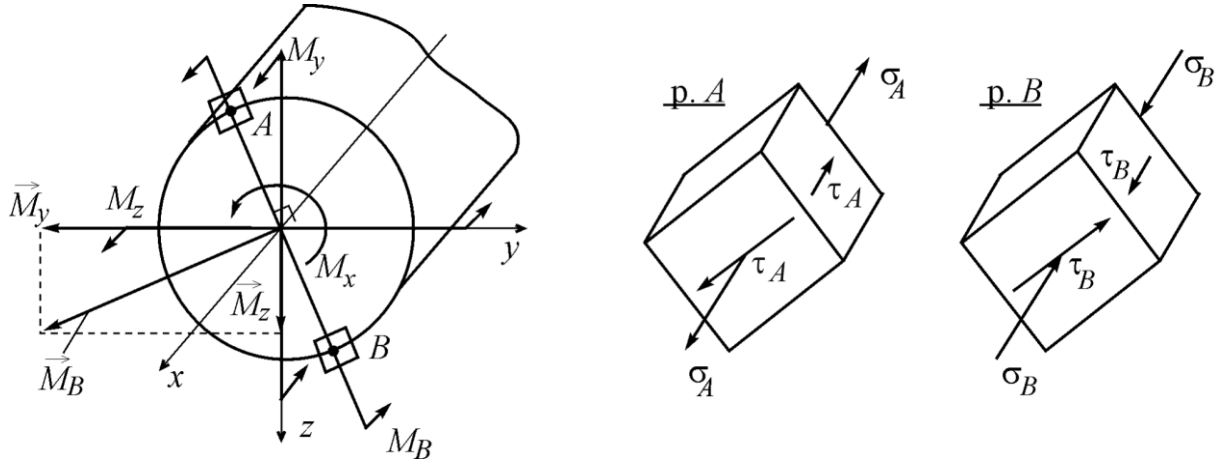### 3 Bending and Torsion of a Circular Cross Section

Fig. 8

Due to polar symmetry of cross-section, determine first of all the plane of resultant bending moment M_B: $M_B = \sqrt{M_y^2 + M_z^2}$.

By analogy with the points A and C for the rectangular section

$$\sigma_{eq_{p.A(p.B)}}^{III} = \sqrt{\left(\frac{M_B}{W_{n.a}}\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \leq [\sigma], \quad W_\rho = 2W_{n.a},$$

$$\sigma_{eq_{p.A(p.B)}}^{III} = \frac{\sqrt{M_x^2 + M_y^2 + M_z^2}}{W_{n.a}},$$

or

$$\sigma_{eq_{p.A(p.B)}}^{III} = \frac{M^{III}}{W_{n.a}} \leq [\sigma] \tag{8}$$

where

$$M^{III} = \sqrt{M_x^2 + M_y^2 + M_z^2} .$$

$$\sigma_{eq_{p.A(p.B)}}^{IV} = \frac{M^{IV}}{W_{n.a}} \leq [\sigma] \tag{9}$$

where
$$M^{IV} = \sqrt{M_y^2 + M_z^2 + \frac{3}{4}M_x^2} \ .$$

4 Bending, Torsion and Tension of a Circular Cross-Section

To determine the diameter of a rod we use formulae (8, 9). For estimating the strength it is necessary to check the strength using the following conditions:

$$\sigma_{eq}^{III} = \sqrt{\left(\left|\frac{N_x}{A}\right| + \left|\frac{M_B}{W_{n.a}}\right|\right)^2 + 4\left(\frac{M_x}{W_\rho}\right)^2} \le [\sigma], \qquad (10)$$

$$\sigma_{eq}^{IV} = \sqrt{\left(\left|\frac{N_x}{A}\right| + \left|\frac{M_B}{W_{n.a}}\right|\right)^2 + 3\left(\frac{M_x}{W_\rho}\right)^2} \le [\sigma]. \qquad (11)$$