LECTURE 19 Combined Loading of a Bar Systems (part 2)
Simplified Cases of Combined Loading

1 Unsymmetrical (Oblique) Bending

By oblique bending is meant a type of bending in which the plane of bending moment does not coincide with any of two principal axes of the section.

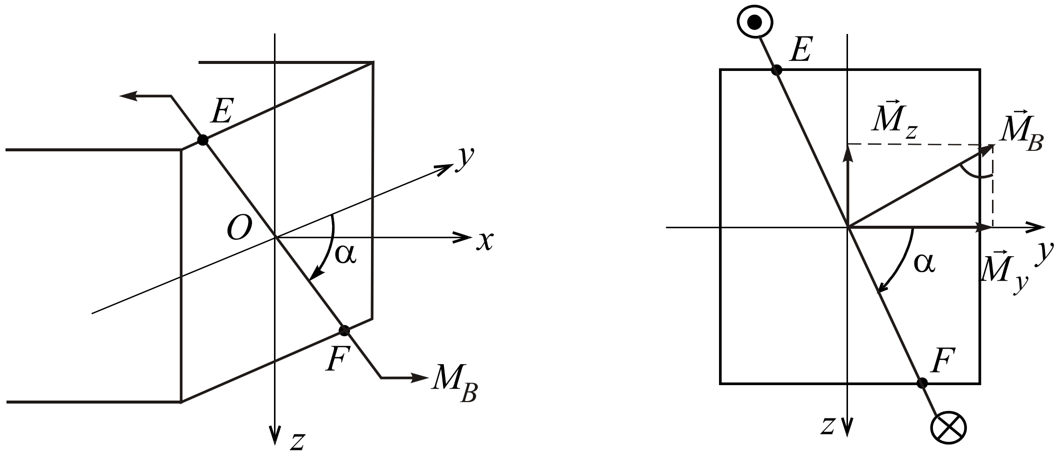

Fig. 1

Oblique bending can most conveniently be treated as two simultaneous plane bendings of a rod in two principal planes $x0z$ and $x0y$ (Fig. 1). For this purpose, the bending moment M_B is resolved into two components about the y and z axes:

$$M_y = M_B \sin\alpha, \quad M_z = M_B \cos\alpha. \tag{1}$$

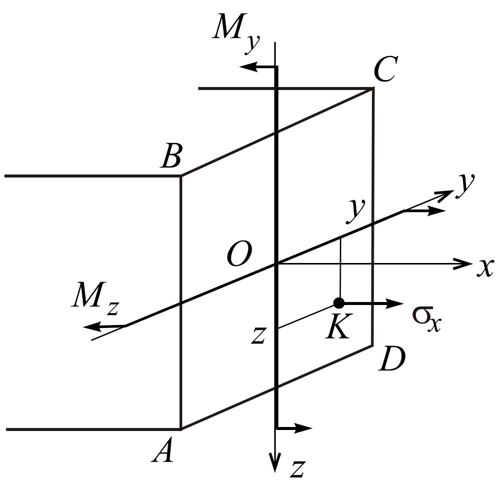

Fig. 2

The normal stress at an arbitrary point K of an arbitrary cross-section $ABCD$ having two co-ordinates z and y is determined as the algebraic sum of the stresses due to the moments M_y and M_z respectively, i.e.

$$\sigma_x = \frac{M_y z}{I_y} + \frac{M_z y}{I_z} \tag{2}$$

or

$$\sigma_x = z\frac{M_B \sin\alpha}{I_y} + y\frac{M_B \cos\alpha}{I_z} = M_B\left(z\frac{\sin\alpha}{I_y} + y\frac{\cos\alpha}{I_z}\right). \tag{3}$$

It is known that centroid *O* belongs to the **neutral axis**, where $\sigma_x = 0$. Its equation may be found analytically and graphically. In the last case we should design scaled graph of σ_x distribution over the contour *ABCD*. Analytically the equation of the neutral axis of the section is found by setting $\sigma_x = 0$ into equation (3):

$$\frac{z}{I_y}\sin\alpha + \frac{y}{I_z}\cos\alpha = 0, \tag{4}$$

or

$$z = \left(-\frac{I_y}{I_z}\cot\alpha\right)y = K_2 y. \tag{5}$$

It can be seen that ***in oblique bending the neutral axis is not perpendicular to the plane of bending moment*** $\boldsymbol{M_B}$. Indeed, the slope K_1 of the trace of the plane of moment (Fig. 1) represents the tangent of the angle α:

$$K_1 = \tan\alpha. \tag{6}$$

The slope of the neutral axis (formula 5) is

$$K_2 = -\frac{I_y}{I_z}\cot\alpha. \tag{7}$$

As $I_z \neq I_y$ in the general case, the condition of perpendicularity of straight lines, known from analytic geometry, is not fulfilled:

$$K_1 \neq -\frac{1}{K_2}. \tag{8}$$

Since the diagram of normal stresses distribution in the section is linear, the maximum stress occurs at the point which is the most remote from the neutral axis. In Fig. 2 two angular points *B* and *D* are most remote with $\sigma_x(M_y)$ and $\sigma_x(M_z)$ summarizing effect:

$$\sigma_D = |\sigma_B| = \sigma_{x_{\max}} = \frac{M_y \times h/2}{I_y} + \frac{M_z \times b/2}{I_z}, \tag{9}$$

where h and b height and width of the section.

Because stress state in B and D points is uniaxial, condition of strength for unsymmetrical bending is

$$\sigma_{x_{\max}} = \frac{M_y \times h/2}{I_y} + \frac{M_z \times b/2}{I_z} = \frac{M_y}{W_y} + \frac{M_z}{W_z} \le [\sigma], \tag{10}$$

Example 1

A rectangular, simply supported beam of length $L = 2\,\text{m}$ is loaded by $F = 10\,\text{kN}$, whose plane of action is inclined at $\alpha = 30°$ to the z axis, as shown in Fig. 3. Determine (a) the maximum bending stress in the beam and (b) the orientation of neural axis.

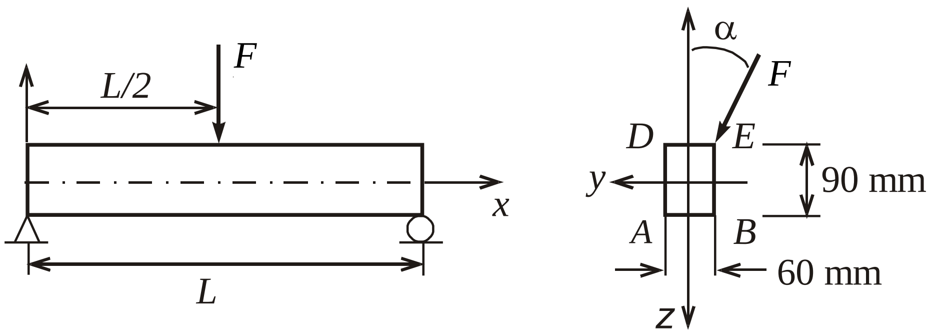

Fig. 3

Solution The **centroidal principal moments of inertia** of the cross section are

$$I_y = \frac{1}{12}(0.06)(0.09)^3 = 3.64 \times 10^{-6}\,\text{m}^4,$$

$$I_z = \frac{1}{12}(0.09)(0.06)^3 = 1.62 \times 10^{-6}\,\text{m}^4.$$

(a) The load components are $F\cos\alpha$ in y-direction and $F\sin\alpha$ in y-direction. Thus, the maximum bending moments occur in the cross section at midspan and are expressed by

$$M_{y_{\max}} = \frac{1}{4}FL\cos\alpha, \quad M_{z_{\max}} = \frac{1}{4}FL\sin\alpha \text{ (see Fig. 4).}$$

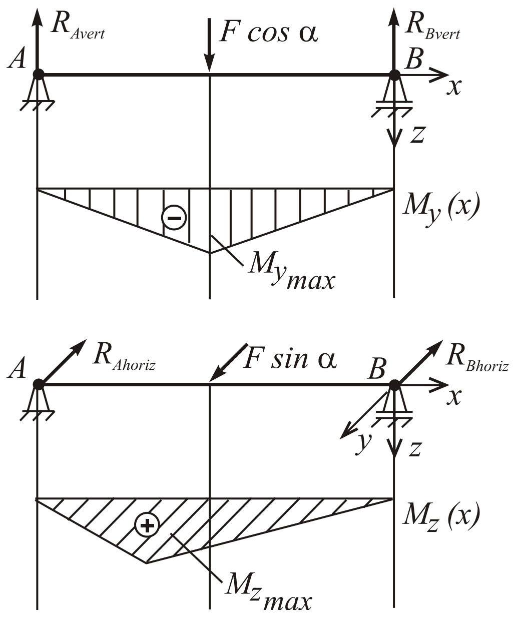

Fig. 4

Substituting the data, we have (Fig. 5):

$$M_{y\max} = \frac{1}{4}(10)(2)\cos 30° = 4.3\,\text{kNm},$$

$$M_{z\max} = \frac{1}{4}(10)(2)\sin 30° = 2.5\,\text{kNm}.$$

To determine the maximum bending stress we must calculate the acting stresses in four angular points of midspan cross section:

$$\sigma_A = \sigma_{\max}\left(M_{y_{\max}}\right) + \sigma_{\max}\left(M_{z_{\max}}\right) =$$

$$= \frac{M_{y_{\max}} z_{\max}}{I_y} + \frac{M_{z_{\max}} y_{\max}}{I_z}.$$

In our case, $z_{\max} = h/2 = 4.5\times10^{-2}$ m, $y_{\max} = b/2 = 3\times10^{-2}$ m.

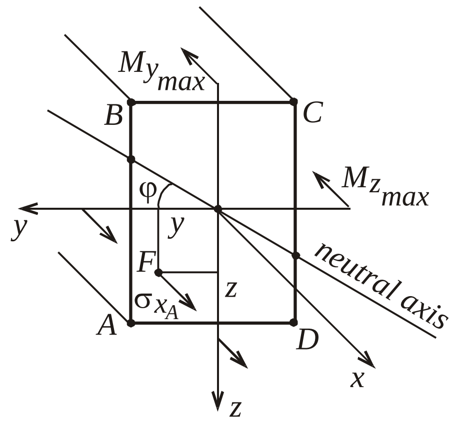

Fig. 5

As the result

$$\sigma_A = \frac{4.3\times10^3\times4.5\times10^{-2}}{3.64\times10^{-6}} + \frac{2.5\times10^3\times3\times10^{-2}}{1.62\times10^{-6}} =$$

$$= +53.2 + 46.3 = +99.5\,\text{MPa},$$

$$\sigma_B = -\sigma_{\max}\left(M_{z_{\max}}\right) + \sigma_{\max}\left(M_{y_{\max}}\right) =$$

$$= -53.2 + 46.3 = -6.9\,\text{MPa},$$

$$\sigma_C = -\sigma_{\max}\left(M_{y_{\max}}\right) - \sigma_{\max}\left(M_{z_{\max}}\right) = -53.2 - 46.3 = -99.5\,\text{MPa},$$

$$\sigma_D = \sigma_{\max}\left(M_{y_{\max}}\right) - \sigma_{\max}\left(M_{z_{\max}}\right) = 53.2 - 46.3 = +6.9\,\text{MPa}.$$

The maximum flexural stress is therefore 99.5 MPa.

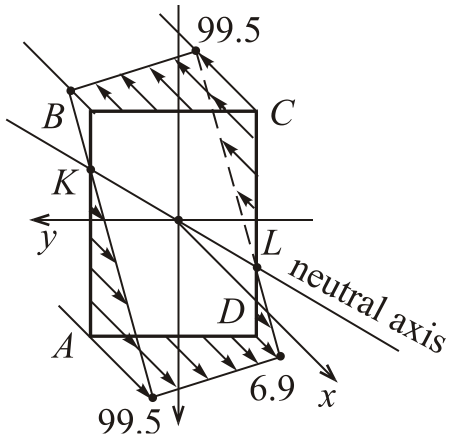

Fig. 6

(b) The angle φ that the neutral surface forms with the horizontal plane is calculated using (3) by analogy:

$$\sigma_F = \frac{M_{y_{\max}}}{I_y} z + \frac{M_{z_{\max}}}{I_z} y = 0,$$

$$z = -\frac{M_{z_{\max}}}{M_{y_{\max}}} \frac{I_y}{I_z} y = -\frac{\sin\alpha}{\cos\alpha} \frac{I_y}{I_z} y = \left(-\frac{I_y}{I_z} \tan\alpha \right) y,$$

$$z = ky, \text{ where } k = \tan\varphi = -\frac{I_y}{I_z} \tan\alpha = -\frac{3.64}{1.62} \tan 30^\circ = -1.297.$$

$$\varphi = -52.4^\circ \text{ (clockwise rotation relative to } y\text{-axis).}$$

The stress distribution across the cross section at the midspan is depicted in Fig. 6.

3 Eccentric Tension and Compression

In eccentric tension the resultant of external forces does not coincide with the axis of a rod, as in ordinary tension, but is shifted relative to the x axis and remains parallel to it (Fig. 7).

Let the point *A* of external force resultant application has the coordinates z_0 and y_0 in the section (Fig. 7). In all cross-sections far from external force application, for example in *KLMN* cross-section (Fig. 8), the moments produced by the resultant force *F* about the principal axes are

$$M_y = F \times z_0, \quad M_z = F \times y_0. \tag{11}$$

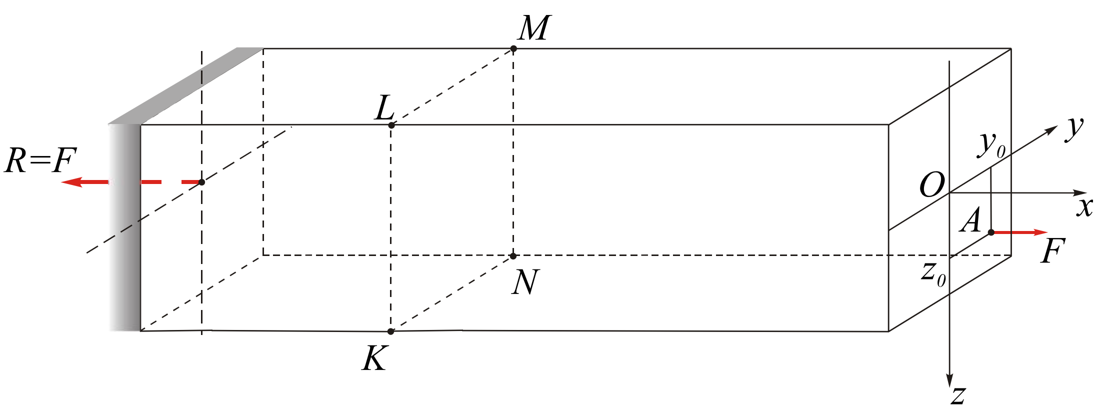

Fig. 7

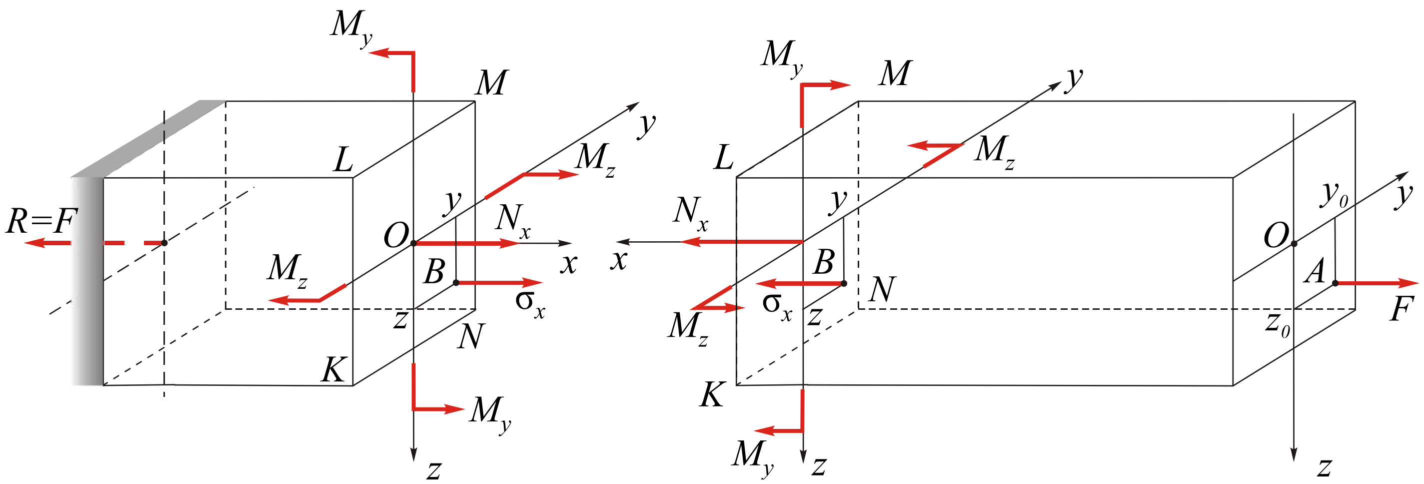

Fig. 8

Thus, eccentric tension-compression is similar to oblique bending. In contrast to the latter, however, ***a normal force*** $N_x = F$***, as well as bending moments, occur in the cross-section of the rod***. The normal stress σ_x at an arbitrary point *B* of *KLMN* section with coordinates *z*, *y* is determined by the following expression:

$$\sigma_x = \frac{N_x}{A} + \frac{M_y}{I_y} z + \frac{M_z}{I_z} y \tag{12}$$

or

$$\left.\begin{aligned} \sigma_x &= \frac{F}{A} + \frac{F \times z_0}{I_y} z + \frac{F \times y_z}{I_z} y, \\ \sigma_x &= \frac{F}{A}\left(1 + \frac{z_0 \times z \times A}{I_y} + \frac{y_z \times y \times A}{I_z}\right), \\ \sigma_x &= \frac{F}{A}\left(1 + \frac{z_0 \times z}{i_y^2} + \frac{y_z \times y}{i_z^2}\right), \end{aligned}\right\} \tag{13}$$

where i_y, i_z denote the radii of gyration of the section

$$i_y = \sqrt{\frac{I_y}{A}}, \quad i_z = \sqrt{\frac{I_z}{A}}. \tag{14}$$

The equation of the neutral axis in the section may be found analytically by setting $\sigma_x = 0$ in equation for stress in arbitrary point (12,13):

$$1+\frac{z_0\times z}{i_y^2}+\frac{y_0\times y}{i_z^2}=0, \quad \rightarrow \quad \frac{z}{\left(-\frac{i_y^2}{z_0}\right)}+\frac{y}{\left(-\frac{i_z^2}{y_0}\right)}=1. \tag{15}$$

This is straight line which does not pass through the centroid of the section. It cuts two segments from the *y* and *z* coordinates

$$a_z=-\frac{i_y^2}{z_0}, \qquad a_y=-\frac{i_z^2}{y_0}. \tag{16}$$

Graphically, the position of neutral axis may be find designing scaled graph of stress distribution along the contour *KLMN*.

The maximum stress occurs in one of four angular points of the cross-section contour where three internal force factors N_x, M_y, M_z produce largest in magnitude stresses. In our case, *N* point is critical for the section and

$$\sigma_N=\sigma_{x_{\max}}=\sigma(N_x)+\sigma_{\max}(M_y)+\sigma_{\max}(M_z)=\frac{N_x}{A}+\frac{M_y}{I_y}h/2+\frac{M_z}{I_z}b/2. \tag{17}$$

Because stress state in critical *N* point is uniaxial condition of strength for eccentric tension is

$$\sigma_{x_{\max}}=\frac{N_x}{A}+\frac{M_y}{I_y}h/2+\frac{M_z}{I_z}b/2=\frac{N_x}{A}+\frac{M_y}{W_y}+\frac{M_z}{W_z}\le[\sigma]. \tag{18}$$

Concept of cross-section core.

In the vicinity of the centroid there exists a region called the **core** of the section. ***If the trace of the force F is inside the core of the section, the stresses are of the same sign at all points in the section***. If the force is applied outside the core of the section, the neutral axis crosses the section and the stresses on the section are both compressive and tensile.

Example 2

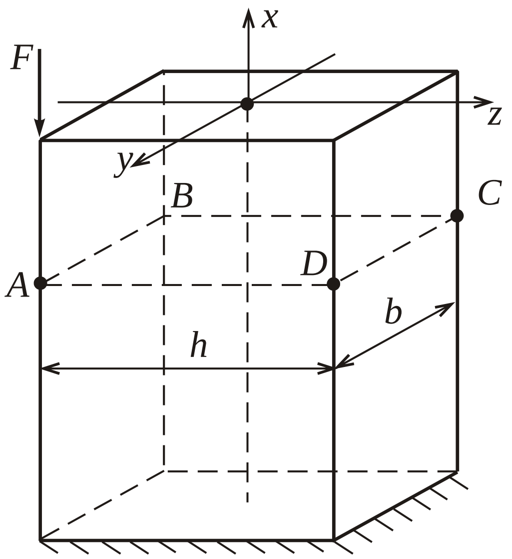

Fig. 9

What is the largest force F that can be supported by block shown in Fig. 8 if the allowable stresses are $[\sigma]_c = 80\,\text{MPa}$ in compression and $[\sigma]_t = 20\,\text{MPa}$ in tension? $h = 0.1\,\text{m}$, $b = 0.06\,\text{m}$.

Solution The eccentric external creates in an arbitrary section *ABCD* a normal internal force N_x and two internal bending moments

$$M_y = F \times \frac{h}{2} = 0.05F \text{ Nm and}$$

$$M_z = F \times \frac{b}{2} = 0.03F \text{ Nm (Fig. 10)}$$

The area and the moments of inertia of the cross section are

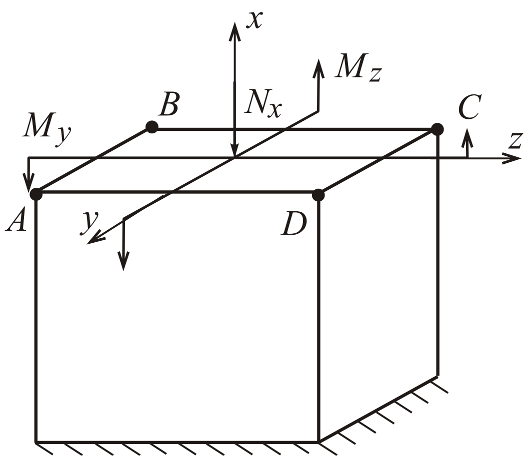

Fig. 10

$$A = 0.06 \times 0.1 = 0.006\,\text{m}^2,$$

$$I_y = (0.06)(0.1)^3 / 12 = 5 \times 10^{-6}\,\text{m}^4,$$

$$I_z = (0.1)(0.06)^3 / 12 = 1.8 \times 10^{-6}\,\text{m}^4.$$

The stress due to the axial load N_x is

$$\sigma(N_x) = -\frac{F}{A} = -\frac{F}{0.006} = -166.67F\ .$$

The maximum values of stress owing to bending moments M_y and M_z are, respectively,

$$\left|\sigma_{\max}(M_y)\right| = \frac{M_y \times z_{\max}}{I_y} = \frac{0.05F(0.05)}{5 \times 10^{-6}} = 500F\ ,$$

$$\left|\sigma_{\max}(M_z)\right| = \frac{M_z \times y_{\max}}{I_z} = \frac{0.03F(0.03)}{1.8 \times 10^{-6}} = 500F\ .$$

The maximum compressive stresses act in the N point

$$\sigma_N = \sigma_{\max} = -\sigma(N_x) - \sigma_{\max}(M_y) - \sigma_{\max}(M_z) = -166.67F - 500F - 500F = -1166.67F$$

.

In another points of the cross-section contour the following stresses act:

$$\sigma_K = -\sigma(N_x) + \sigma_{\max}(M_y) - \sigma_{\max}(M_z) = -166.67F + 500F - 500F = -166.67F\ ,$$

$$\sigma_L = -\sigma(N_x) - \sigma_{\max}(M_y) - \sigma_{\max}(M_z) = -166.67F - 500F + 500F = +833.33F\ ,$$

$$\sigma_D = -\sigma(N_x) - \sigma_{\max}(M_y) - \sigma_{\max}(M_z) = -166.67F - 500F + 500F = -166.67F\ .$$

The largest, so called allowable external force $[F]$, may be calculated from the condition of strength in the most tensioned and compressed points of cross section

$$\underset{\max}{\sigma_{tens}} = \sigma_C \le [\sigma]_t, \quad 833.33[F]_{tens} = 20\times10^6,$$

$$[F]_{tens} = \frac{20\times10^6}{833.33} = 24\,\text{kN},$$

$$\left|\underset{\max}{\sigma_{comp}}\right| = |\sigma_A| \le [\sigma]_C, \quad 1166.67.33[F]_{comp} = 80\times10^6,$$

$$[F]_{comp} = \frac{80\times10^6}{1166.67} = 68.6\,\text{kN}.$$

The magnitude of the maximum allowable eccentric external load is the smaller of these values, $[F] = 24\,\text{kN}$.