LECTURE 20 Generalized Forces and Generalized Displacements. Strain Energy of a Rod in General Case of Loading

Generalized force F_n ***is a force factor in general (F, q, M,…). Generalized displacement*** δ_n ***is a geometrical parameter on which the generalized force*** F_n ***does work.*** If, for example, F_n is understood to be an external bending moment M in plane bending deformation, then δ_n represents an angular or linear displacement of selected cross-section in the plane of moment action.

1 Displacements of a Rod under Arbitrary Loading

1.1 The work done by external elastic force

(A) The work done by a concentrated force:

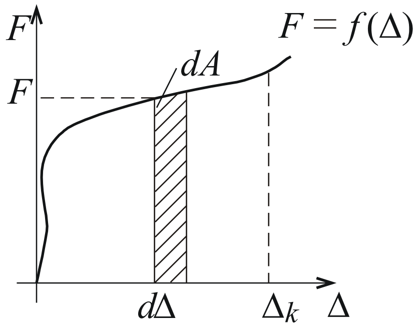

Fig. 1

In general case, the **elementary work** dA is equal to $Fd\Delta$, i.e.

$$dA = Fd\Delta. \qquad (1)$$

By summing these quantities we obtain

$$A = \int_0^{\Delta_k} Fd\Delta. \qquad (2)$$

If $F = c\Delta$ (within elasticity limitations (Fig. 2)), then

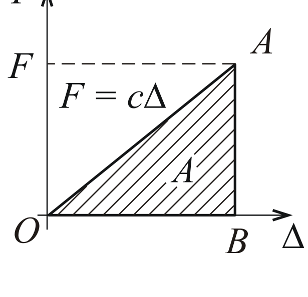

Fig. 2

$$A = \int_0^{\Delta} c\Delta d\Delta = c\frac{\Delta^2}{2}\bigg|_0^{\Delta} = \frac{c\Delta^2}{2} = \frac{c\Delta\Delta}{2} = \frac{F\Delta}{2}. \tag{3}$$

Conclusion: the work done by elastic concentrated force is equal to the area of the triangle OAB.

(B) The work done by a couple of elastic forces:

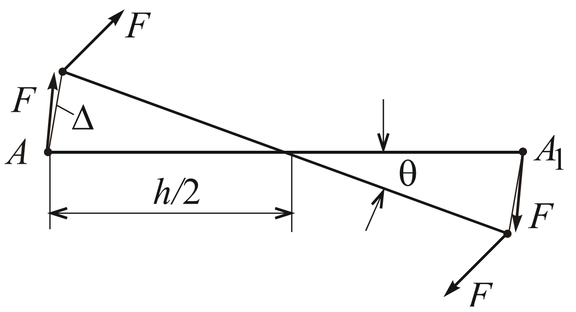

Fig. 3

The displacements of the points A and A_1 produced by the couple of forces are

$$\Delta = \frac{h}{2}\theta. \tag{4}$$

The work done by a couple of forces is

$$A = 2\frac{F\Delta}{2} = 2\frac{Fh\theta}{4} = \frac{Fh\theta}{2}. \tag{5}$$

But $Fh = M$, then

$$A = \frac{M\theta}{2}. \tag{6}$$

1.2 The work done by internal forces. Strain energy and strain energy density

It was mentioned above, that internal forces really represent stresses acting at any point of elastic solid (see Lecture 4).

It was concluded earlier, that within elasticity limitations, the work done by internal forces is equal to potential energy of strain (strain energy) stored in the element: $U = A$.

(A) Determining the potential energy of strain in tension (compression)

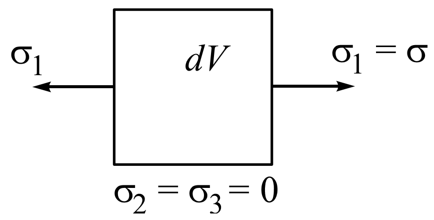

Fig. 5

It is known (see Eq. 34 in Lecture 9) that strain energy accumulated in infinitely small volume $dV = dxdydz$ is

$$dU = U_0 dV = \frac{dV}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu\left(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3\right)\right],$$

where U_0 – **strain energy density** (G/m^3).

In our case (tension-compression), $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$. As a result,

$$dU = \frac{\sigma^2}{2E} dV \, . \qquad (7)$$

Then strain energy density

$$U_0 = \frac{\sigma^2}{2E} \, . \qquad (8)$$

(B) Determining the potential energy of strain in pure shear

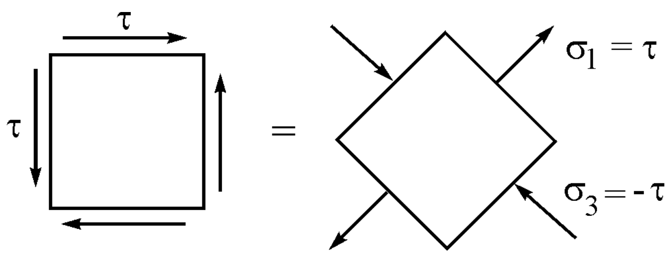

Fig. 6

$$dU = U_0 dV = \frac{1}{2E}\left[\tau^2 + \tau^2 - 2\mu(-\tau)\times\tau\right]dV = 2\frac{dV}{2E}\tau^2\left(1+\mu\right) =$$

$$=\frac{\tau^2}{2}\frac{dV}{\frac{E}{2(1+\mu)}}=\left\{\frac{E}{2(1+\mu)}=G\right\}=\frac{\tau^2}{2G}dV\,, \tag{9}$$

In this case (pure shear deformation),

$$U_0=\frac{\tau^2}{2G}\,. \tag{10}$$

It should be noted that ***within the elementary volume of solid its stress state is homogeneous***.

(C) Determining the strain energy of a rod in general case of loading.

Let us select in elastic rod under combined loading its elementary volume denoted earlier as stress element ($dV = dA \times dx$) isolating an elementary portion of length dx from a rod:

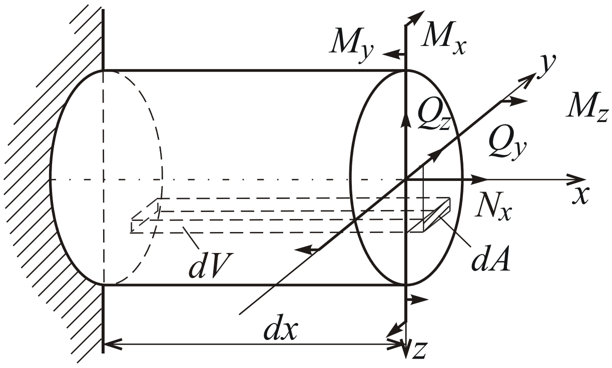

Fig.7

In general, six internal force factors occur at each cross section of the rod: three moments and three forces (M_x, M_y, M_z and N_x, Q_z, Q_y). The potential energy of the stress element dV may be regarded as the sum of work done by each of the six force factors acting separately

$$dA = dU = dU(N_x)+dU(Q_y)+dU(Q_z)+dU(M_x)+dU(M_y)+dU(M_z)\,. \tag{11}$$

As it is known, strain energy density

$$U_0=\frac{dU}{dV}\to dU=U_0 dV=\left(\int_A U_0 dA\right)dx. \tag{12}$$

Then

$$dU(N_x)=\left[\int_A U_0(N_x)dA\right]dx=\left(\int_A \frac{\sigma^2}{2E}dA\right)dx=\left\{\sigma=\frac{N_x}{A}\right\}=\int_A \frac{N_x^2 dA}{2EA^2}dx=\frac{N_x^2}{2EA}dx. \tag{13}$$

$$dU(Q_z)=\left[\int_A U_0(Q_z)dA\right]dx=\left(\int_A \frac{\tau^2}{2G}dA\right)dx=\left\{\tau=\frac{Q_z S_y^*}{I_y b(z)}\right\}=$$

$$\left(\int_A \frac{Q_z^2 S_y^{*2}}{2GI_y^2[b(z)]^2}dA\right)dx=\frac{Q_z^2}{2GA}\underbrace{\left(\frac{A}{I_y^2}\int_A \frac{S_y^{*2}}{[b(z)]^2}dA\right)}_{K_z}dx=K_z\frac{Q_z^2}{2GA}dx. \tag{14}$$

By analogy:

$$dU(Q_y)=K_y\frac{Q_y^2}{2GA}dx. \tag{15}$$

$$dU(M_x)=\left\{\int_A U_0(M_x)dA\right\}dx=\left(\int_A \frac{\tau^2}{2G}dA\right)dx=\left\{\tau=\frac{M_x\rho}{I_\rho}\right\}=$$

$$=\left(\int_A \frac{M_x^2\rho^2}{2GI_\rho^2}dA\right)dx=\frac{M_x^2}{2GI_\rho^2}\left(\int_A \rho^2 dA\right)dx=\left\{\int_A \rho^2 dA=I_\rho\right\}=\frac{M_x^2}{2GI_\rho}dx.$$

$$dU(M_x)=\frac{M_x^2}{2GI_\rho}dx. \tag{16}$$

$$dU(M_y)=\left(\int_A U_0(M_y)dA\right)dx=\left(\int_A \frac{\sigma^2}{2E}dA\right)dx=\left\{\sigma=\frac{M_y z}{I_y}\right\}=$$

$$=\left(\int_A \frac{M_y^2 z^2}{2EI_y^2}dA\right)dx=\frac{M_y{}^2}{2EI_y}dx. \tag{17}$$

By analogy,

$$dU\left(M_z\right)=\frac{M_z^2}{2EI_z}dx. \tag{18}$$

Now expression (9) becomes

$$dU=\frac{N_x^2dx}{2EA}+K_y\frac{Q_y^2dx}{2GA}+K_z\frac{Q_z^2dx}{2GA}+\frac{M_x^2dx}{2GI_p}+\frac{M_y^2dx}{2EI_y}+\frac{M_z^2dx}{2EI_z}. \tag{19}$$

In order to obtain the potential energy of the whole rod this expression must be integrated over the length of the rod

$$U=\int_l\frac{N_x^2}{2EA}dx+\int_l\frac{K_yQ_y^2}{2GA}dx+\int_l\frac{K_zQ_z^2}{2GA}dx+\int_l\frac{M_x^2}{2GI_\rho}dx+\int_l\frac{M_y^2}{2EI_y}dx+\int_l\frac{M_z^2}{2EI_z}dx. \tag{20}$$

All the terms in expression (18) are not always equivalent. In the great majority of systems encountered in practice with components acting in bending or torsion, the last three terms in expression (18) are less appreciable than the first three. In other words, the ***energy due to tension and shear is considerably less then the energy due to bending and torsion***.