LECTURE 21 Mohr's Method for Calculation of Generalized Displacements

1 The Reciprocal Theorem

The reciprocal theorem is one of the general theorems in mechanics of materials. It follows directly from the principle of superposition and applies to all systems for which this principle is valid.

Definition ***The work done by the first generalized force (F_1) on the displacement of its point of application produced by the second generalized force (F_2) is equal to the work done by the second force on the displacement of its point of application produced by the first force.***

Let us apply two generalized forces F_1 and F_2 to elastic beam in *A* and *B* points, respectively, and analyze corresponding deflections Δ_1, Δ_2 (see Fig. 1).

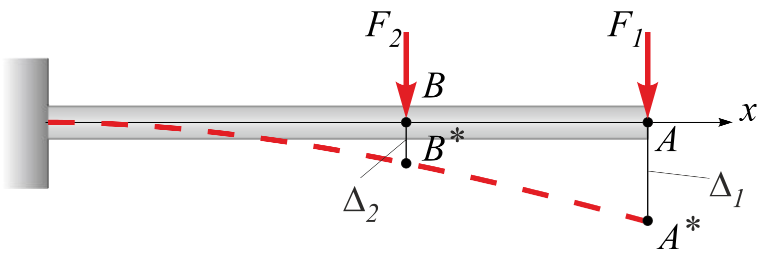


Fig. 1

Obviously,

$$\begin{aligned} AA^* &= \Delta_1 = \Delta_1(F_1) + \Delta_1(F_2) = \delta_{11} + \delta_{22}, \\ BB^* &= \Delta_2 = \Delta_2(F_1) + \Delta_2(F_2) = \delta_{21} + \delta_{22}, \end{aligned} \tag{1}$$

where δ_{ik} is the displacement of the point "*i*" in the direction of the force "*i*" caused by the force "*k*".

Consider two possible ways of F_1 and F_2 forces application.

Way 1 ***We first apply the force F_1 at the point A (see Fig. 2).***

This force does the work (see Fig. 3, left) on the displacement $AA_1 = \delta_{11}$:

$$W_{11} = \frac{1}{2} F_1 \delta_{11}. \tag{2}$$

Further we apply the force F_2 ***at the point B***. This force does work on displacement $B_1B_2 = \delta_{22}$ which is expressed similarly by Fig. 3, right:

$$W_{22} = \frac{1}{2} F_2 \delta_{22}. \tag{3}$$

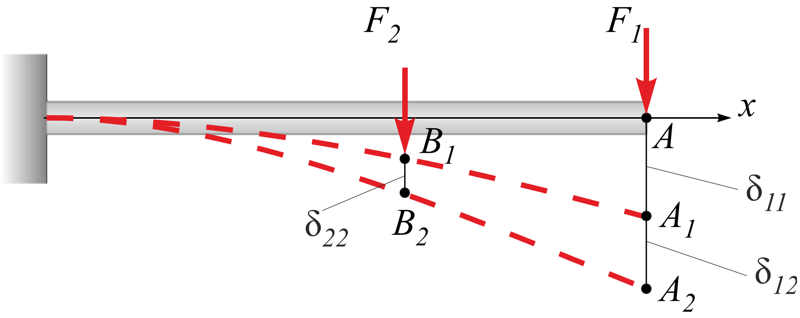


Fig. 2

At the same time the force F_1 does work, too, since the application of the force F_2 causes a displacement of the point A ($A_1A_2 = \delta_{12}$):

$$W_{12} = F_1 \delta_{12}. \tag{4}$$

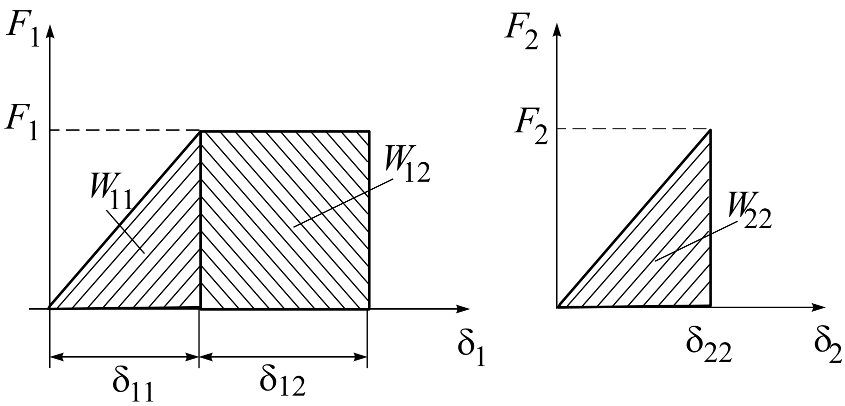


Fig. 3

By summation we obtain the total work done by the forces when they are applied in direct order

$$W = W_{11} + W_{22} + W_{12},$$

$$W = \frac{1}{2} F_1 \delta_{11} + \frac{1}{2} F_2 \delta_{22} + F_1 \delta_{12}. \tag{5}$$

Way 2 ***We apply first the force*** F_2 ***and then*** F_1 ***(see Fig. 4).***

Force F_2 does work on the displacement $BB_1 = \delta_{22}$. Following force F_1 does work on the displacement $A_1A_2 = \delta_{11}$. Simultaneously, the force F_2 does complementary work too because force F_1 application causes the displacement of B point ($B_1B_2 = \delta_{21}$):

$$W_{21} = F_2\delta_{21}. \tag{6}$$

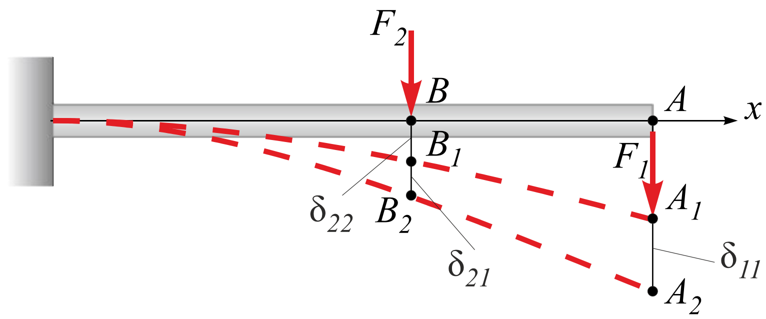


Fig. 4

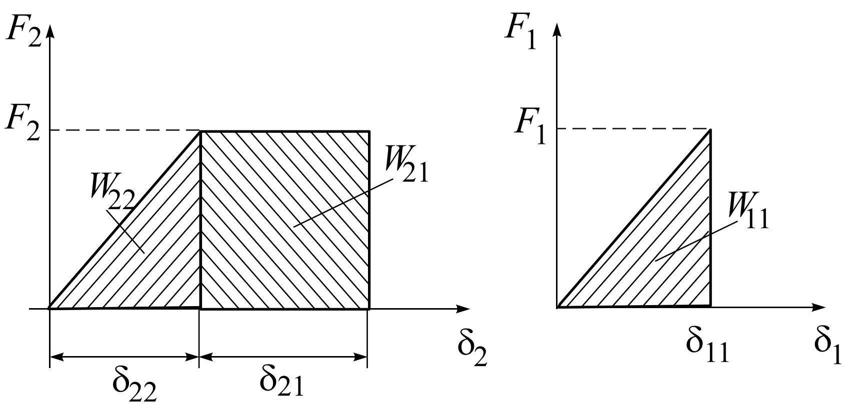


Fig. 5

Obviously, total work is expressed as

$$W = W_{11} + W_{21} + W_{22} = \frac{1}{2}F_1\delta_{11} + F_2\delta_{21} + \frac{1}{2}F_2\delta_{22}. \tag{7}$$

By equating the work, we find

$$F_1\delta_{12} = F_2\delta_{21}. \tag{8}$$

Sometimes the reciprocal theorem is interpreted in more narrow sense. When $F_1 = F_2 = F$ expression (8) becomes

$$\delta_{12} = \delta_{21}, \tag{9}$$

and the displacement of the point A produced by the force applied at the point B is equal to the displacement of the point B produced by the same force but applied at the point A.

2 Mohr's Integral. Determination of Generalized Displacements

Let us consider an elastic solid subjected to an arbitrary system of forces F_1 and F_2 (F_1 and F_2 are generalized forces).

Let us assume that we apply the force F_1. It causes the internal force factors N_{x_1}, Q_{y_1}, Q_{z_1}, M_{x_1}, M_{y_1}, M_{z_1} in cross-sections of the rod.

Set up the expression for the potential energy of strain

$$W_{11}=U_1=\sum_{i=1}^{n}\left(\int_{l_i}\frac{N_{x_1}^2}{2EA}dx+\int_{l_i}K_y\frac{Q_{y_1}^2}{2GA}dx+\int_{l_i}K_z\frac{Q_{z_1}^2}{2GA}dx+\right.$$

$$\left.+\int_{l_i}\frac{M_{x_1}^2}{2GI_\rho}dx+\int_{l_i}\frac{M_{y_1}^2}{2EI_y}dx+\int_{l_i}\frac{M_{z_1}^2}{2EI_z}dx\right). \qquad (10)$$

Let us assume, that we apply the force F_2. Then

$$W_{22}=U_2=\sum_{i=1}^{n}\left(\int_{l_i}\frac{N_{x_2}^2}{2EA}dx+\int_{l_i}K_y\frac{Q_{y_2}^2}{2GA}dx+\int_{l_i}K_z\frac{Q_{z_2}^2}{2GA}dx+\right.$$

$$\left.+\int_{l_i}\frac{M_{x_2}^2}{2GI_\rho}dx+\int_{l_i}\frac{M_{y_2}^2}{2EI_y}dx+\int_{l_i}\frac{M_{z_2}^2}{2EI_z}dx\right). \qquad (11)$$

Further ***we apply the forces F_1 and F_2 simultaneously***:

$$W=U=\sum_{i=1}^{n}\left(\int_{l_i}\frac{\left(N_{x_1}+N_{x_2}\right)^2}{2EA}dx+\int_{l_i}K_y\frac{\left(Q_{y_1}+Q_{y_2}\right)^2}{2GA}dx+\int_{l_i}K_z\frac{\left(Q_{z_1}+Q_{z_2}\right)^2}{2GA}dx+\right.$$

$$+\int_{l_i}\frac{\left(M_{x_1}+M_{x_2}\right)^2}{2GI_\rho}dx+\int_{l_i}\frac{\left(M_{y_1}+M_{y_2}\right)^2}{2EI_y}dx+\int_{l_i}\frac{\left(M_{z_1}+M_{z_2}\right)^2}{2EI_z}dx\Bigg). \qquad (12)$$

According to formula (7)

$$W=W_{11}+W_{21}+W_{22}\rightarrow W_{21}=W-W_{11}-W_{22}\text{, or}$$

$$W_{21}=F_2\delta_{21}=\sum_{i=1}^{n}\left(\int_l\frac{\left(N_{x_1}+N_{x_2}\right)^2dx}{2EA}+\ldots+\int_l\frac{\left(M_{z_1}+M_{z_2}\right)^2dx}{2EI_z}\right)-\sum_{i=1}^{n}\left(\int_l\frac{N_{x_1}{}^2dx}{2EA}+\ldots+\int_l\frac{M_{z_1}{}^2dx}{2EI_z}\right)-$$

$$-\sum_{i=1}^{n}\left(\int_l\frac{N_{x_2}{}^2dx}{2EA}+\ldots+\int_l\frac{M_{z_2}{}^2dx}{2EI_z}\right)=\sum_{i=1}^{n}\left(\int_l\frac{N_{x_1}N_{x_2}}{EA}dx+\ldots+\int_l\frac{M_{z_1}M_{z_2}}{EI_z}dx\right).$$

Let us now assume, that $F_2=1$ and denote $F_1=F$. Then

$$W_{21}=\delta_{21}\times1=\delta_{21}=\delta=\sum_{i=1}^{n}\left(\int_{l_i}\frac{N_{x_F}\bar{N}_x}{EA}dx+...+\int_{l_i}\frac{M_{z_F}\bar{M}_z}{EI_z}dx\right), \qquad (13)$$

where $\bar{N}_x$, $\bar{Q}_y$, $\bar{Q}_z$, $\bar{M}_x$, $\bar{M}_y$, $\bar{M}_z$ ***are the internal force factors caused by the unit force***. Dividing both parts of Eq. (13) by unit force we get in the left part of (13) δ_{21} ***as the displacement of point of $F_2=1$ application under the action of $F_1=F$ force***. The integrals obtained are known as **Mohr's integrals** for calculation of displacements. ***Bending displacements are most significant in the rod*** under consideration. Displacements due to tension and shear are as small in relation to bending displacements as the energy due to tension and shear in relation to energy due to bending.

Hence of six Mohr's integrals (13) we take the one for bending:

$$\delta=\sum_{i=1}^{n}\int_{l_i}\frac{M_{y_F}(x)\bar{M}_y(x)}{EI_y}dx. \qquad (14)$$

Example 1

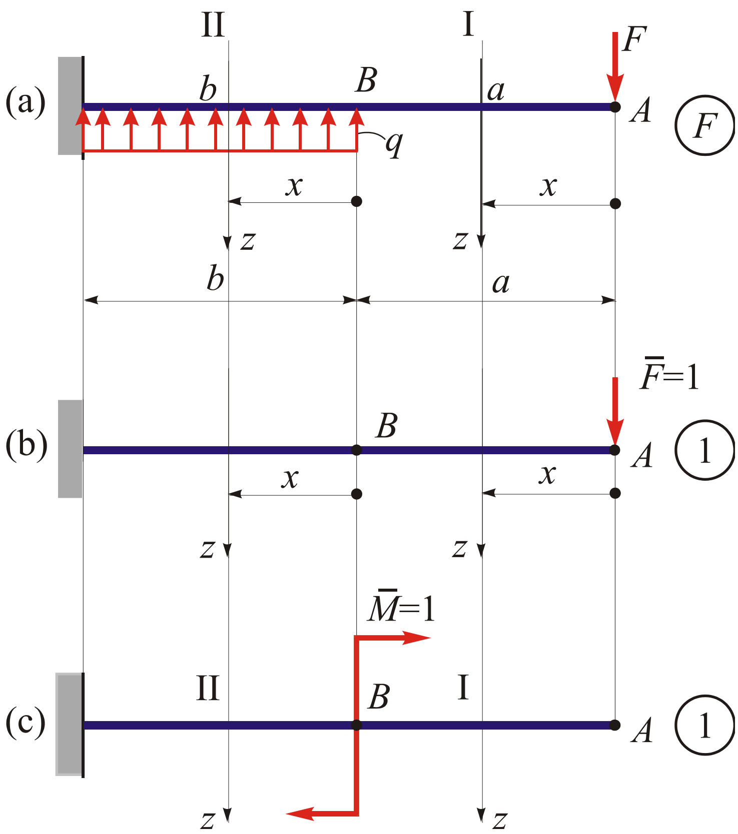


Fig. 6

Given: $F = 10\,\text{kN}$, $q = 60\,\text{kN/m}$, $a = 2\,\text{m}$, $b = 3\,\text{m}$, $EI_y = const$.

It is necessary to determine the vertical displacement (deflection) of the point A and slope of the B cross-section.

1. We have to determine the vertical displacement of the point A. For this purpose, we must apply at this point the ***unit dimensionless force*** $\boldsymbol{\overline{F} = 1}$ ***in vertical direction*** and design corresponding unit system (see Fig. 6).

2. We set up the expressions for the bending moments for each portion, taking the same portions for the force (a) and unit (b) systems:

$$M_{yF}^{I}(x) = -Fx = -10x,$$

$$\overline{M}_{y}^{I}(x) = -1x = -1x,$$

$$M_{y_F}^{II}(x) = -F(x+a) + \frac{qx^2}{2} = -10(x+2) + \frac{60x^2}{2} = -20 - 10x + 30x^2,$$

$$\overline{M}_{y}^{II}(x) = -1(2+x) = -2 - x.$$

3. We substitute the bending moments in expression (14):

$$\delta_A = \frac{1}{EI}\left(\int_0^2 (-10x)(-x)\,dx + \int_0^3 \left(-20 - 10x + 30x^2\right)(-2 - x)\,dx\right) = -\frac{730.8}{EI}.$$

4. ***The minus indicates that the displacement of the point A is not in the direction of the unit force but is opposite to it.***

5. For determining the angular displacement (angle of rotation, slope) of B section we should apply the ***unit dimensionless moment*** $\overline{M}=1$ at this point (see Fig. 6).

6. We set up the expressions for the bending moments for each portion, taking the same portions for the force (a) and unit (c) systems (see Fig. 6):

$$I-I:\ 0<x<a$$

$$M_{yF}^{I}(x)=-Fx=-10x,$$

$$\overline{M}_{y}^{I}(x)=0,$$

$$II-II:\ 0<x<b$$

$$M_{y_F}^{II}(x)=-F(x+a)+\frac{qx^2}{2}=-10(x+2)+\frac{60x^2}{2}=-20-10x+30x^2,$$

$$\overline{M}_{y}^{II}(x)=+1.$$

7. We substitute the bending moments in expression (14):

$$\theta_B=\frac{1}{EI}\left(\int_0^2(-10x)(0)dx+\int_0^2\left(-20-10x+30x^2\right)(+1)dx\right)=+\frac{165}{EI}.$$

Note. The plus sign indicates that B section is turned out counterclockwise

3 Vereshchagin's Method for Graphical Solution of Mohr's Integral

Suppose, for example, it is necessary to take the integral of the product of two functions $f_1(x)\times f_2(x)$ in the portion of length l:

$$I=\int_0^l f_1(x)\times f_2(x)dx. \tag{15}$$

Note, that at least one of these functions is linear.

Let $f_2(x)=b+kx$. Expression (15) then becomes

$$I=b\int_0^l f_1(x)dx+k\int_0^l f_1(x)xdx. \tag{16}$$

The first of the integrals above represents the area bordered by the curve $f_1(x)$ (the area under the $f_1(x)$ graph on Fig. 7):

$$\int_0^l f_1(x)dx = \omega_1. \tag{17}$$

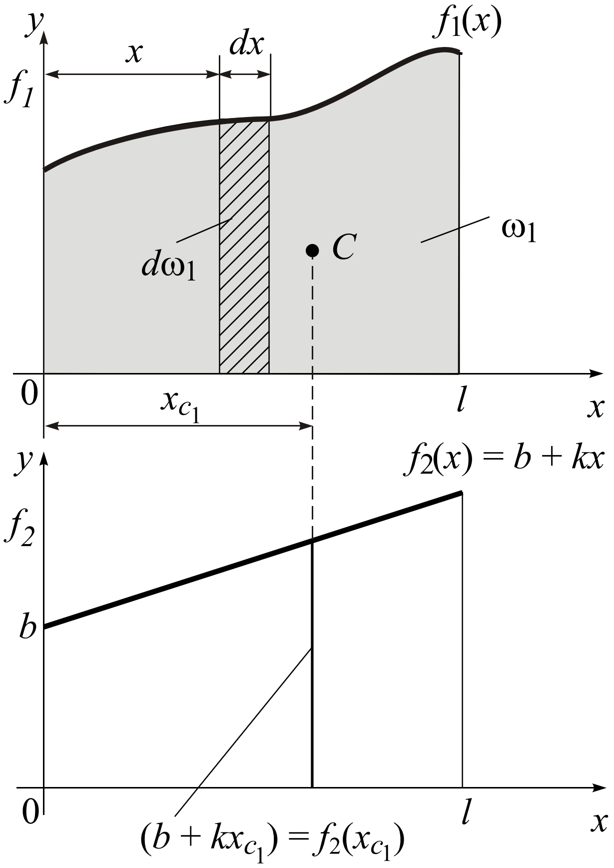


Fig. 7

The second integral represents the static moment (first moment) of this area with respect to the *y* axis, i.e.

$$\int_0^l f_1(x)xdx = S_y = \omega_1 x_c, \tag{18}$$

where x_c is the **co-ordinate of the centroid** of the first diagram. We now obtain

$$I = \omega_1\left(b + kx_c\right). \tag{19}$$

But

$$b + kx_c = f_2\left(x_c\right). \tag{20}$$

Consequently,

$$I = \omega_1 \times f_2(x_c). \tag{21}$$

Thus, by **Vereshchagin's method** ***integration is replaced by multiplication of the area under the first diagram by the ordinate of the second (linear) diagram directly below the centroid of the first one***.

Each of Mohr's integrals (14) includes the product of functions. Vereshchagin's method is applicable to any of the six integrals.

For plane bending deformation, one of six Mohr's integrals allows deflection and slope (cross-sectional rotation) calculation according to equation (14):

$$\delta = \sum_{i=1}^{n} \int_{l_i} \frac{M_{y_F}(x)\bar{M}_y(x)}{EI_y} dx = \sum_{i=1}^{n} \frac{\omega_i \eta_i}{EI_y}, \tag{22}$$

where ω_i – area of the $M_{y_{F_i}}$ diagram for particular portion,

η_i – ordinate of $\bar{M}_y(x)$ diagram under $M_{y_{F_i}}$ centroid.

In cases both functions $f_1(x)$ and $f_2(x)$ are linear the operation of multiplication has the commutative property. In such cases the area under the first diagram should be multiplied by the ordinate of the second diagram or the area under the second diagram should be multiplied by the ordinate of the first diagram.

Note. Graphical method is preferable when the graph $M_{yF}(x)$ ***is boarded by straight lines and allows simple calculation of*** ω_i**.**

Example 2

Given: $F = 10\,\text{kN}$, $m = 60\,\text{kNm}$, $a = 2\,\text{m}$, $b = 3\,\text{m}$, $EI_y = const$.

It is necessary to determine graphically the vertical displacement (deflection) of the point *A* and slope of the *B* cross-section (see Fig. 8).

1. We have to determine the vertical displacement of the point *A*. For this purpose, we must apply at this point the ***unit dimensionless force*** $\bar{F} = 1$ ***in vertical direction*** and design corresponding unit system (see Fig. 8).

To design the diagrams of $M_{yF}(x)$ and $\bar{M}_y(x)$ functions, we set up the expressions for the bending moments for each portion, taking the same portions for the force (a) and unit (b) systems:

$$I-I \quad 0<x<a$$

$$M^I_{yF}(x)=-Fx=-10x\Big|_{x=0}=0\Big|_{x=2}=-20\,\text{kNm},$$

$$\bar{M}^I_y(x)=-1x\Big|_{x=0}=0\Big|_{x=2}=-2\,\text{m},$$

$$II-II \quad 0<x<b$$

$$M^{II}_{y_F}(x)=-F(x+a)+\frac{qx^2}{2}=-10(x+2)+\frac{60x^2}{2}=$$

$$=-20-10x+30x^2\Big|_{x=0}=40\Big|_{x=3}=10\,\text{kNm},$$

$$\overline{M}^{II}_y(x)=-1(a+x)=-2-x\Big|_{x=0}=-2\Big|_{x=3}=-5\,\text{m}.$$

Corresponding graphs of bending moments $M_{y_F}(x)$ and $\bar{M}_y(x)$ are designed and illustrated on Fig. 8 a,b to apply the formula for graphical calculation of the Mohr's integral:

$$\delta=\sum_{i=1}^{n}\frac{\omega_i\eta_i}{EI_y},$$

where ω_i – area of the $M_{y_{F_i}}$ diagram for particular portion,

η_i – ordinate of $\bar{M}_y(x)$ diagram under $M_{y_{F_i}}$ centroid.

In our case of δ_A calculation,

$$\omega_1=-\frac{(20\times 2)}{2}=-20\,\text{kNm}^2,\qquad \eta_1=-\frac{2}{3}\times 2=-1.33\,\text{m},\qquad \omega_2=+(10\times 3)=+30\,\text{kNm}^2,$$

$$\eta_2=-3.5\,\text{m},\ \omega_3=+\frac{(30\times 3)}{2}=+45\ \text{kNm}^2,\ \eta_3=-\left(2+\frac{1}{3}\times 3\right)=-3\,\text{m}.$$

$$\delta_A=\frac{1}{EI_y}\left(\omega_1\eta_1+\omega_2\eta_2+\omega_3\eta_3\right)=\frac{1}{EI_y}\left[(-20)(-1.33)+(+30)(-3.5)+(45)(-3)\right]=$$

$$=\frac{1}{EI_y}\left[+26.6-105-135\right]=-\frac{213.4}{EI_y}.$$

Note: the minus indicates that the displacement of the point A is not in the direction of the unit force but is opposite to it.

3. For determining the angular displacement (angle of rotation, slope) of B section we should apply the ***unit dimensionless moment*** $\overline{M}=1$ at this point (see Fig. 8c).

We set up the expressions for the bending moments for each portion, taking the same portions for the force (a) and unit (c) systems (see Fig. 8):

$$I-I \quad 0<x<a$$

$$M_{yF}^{I}(x)=-Fx=-10x,$$

$$\bar{M}_{y}^{I}(x)=0,$$

$$II-II \quad 0<x<b$$

$$M_{y_F}^{II}(x)=-F(x+a)+\frac{qx^2}{2}=-10(x+2)+\frac{60x^2}{2}=-20-10x+30x^2,$$

$$\overline{M}_{y}^{II}(x)=-1, \text{ dimensionless.}$$

Corresponding graphs of bending moments $M_{y_F}(x)$ and $\bar{M}_y(x)$ are designed and illustrated on Fig. 8a,c to apply the formula for graphical calculation of the Mohr's integral:

In our case of δ_A calculation,

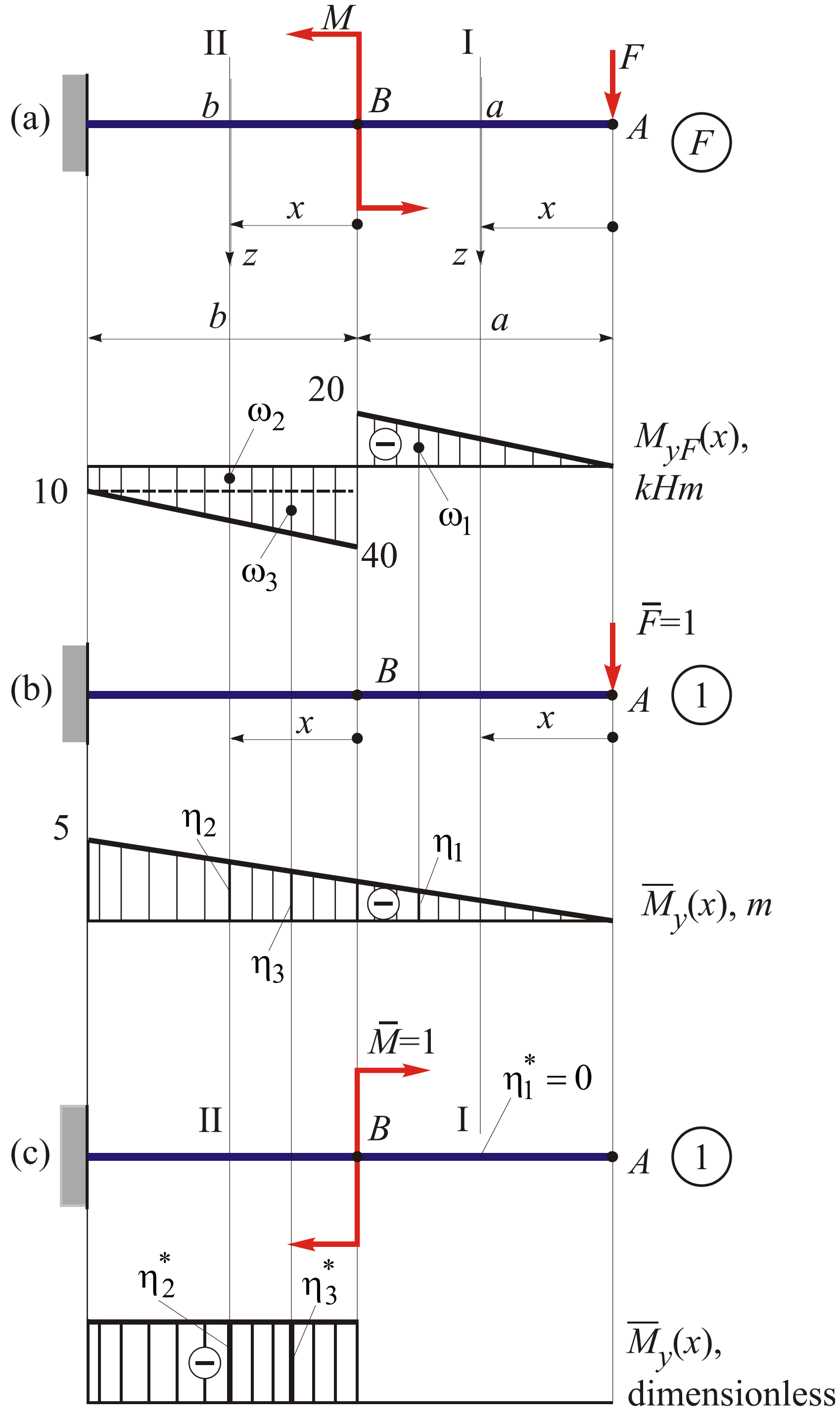
$\omega_1=-\dfrac{(20\times 2)}{2}=-20\,\text{kNm}^2,\qquad \eta_1{*}=0\,\text{m},\qquad \omega_2=+(10\times 3)=+30\,\text{kNm}^2,\qquad \eta_2{*}=-1,$

dimensionless, $\omega_3=+\dfrac{(30\times 3)}{2}=+45\,\text{kNm}^2$, $\eta_3{*}=-1$, dimensionless .

$$\theta_B=\frac{1}{EI_y}\left(\omega_1\eta_1{*}+\omega_2\eta_2{*}+\omega_3\eta_3{*}\right)=\frac{1}{EI_y}\left[(-20)(0)+(+30)(-1)+(45)(-1)\right]=$$

$$=\frac{1}{EI_y}\left[0-30-45\right]=-\frac{75}{EI_y}.$$

Note. The minus sign indicates that B section is turned out counterclockwise



M
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MyF(x), kHm
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ω1
40
ω3
F=1
(b)
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A
1
x
x
5
η2
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My(x), m
η3
M=1
η1* = 0
II
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η3*
My(x), dimensionless

Fig. 8