LECTURE 22 Analysis of Statically Indeterminate Bar Systems by a Force Method

1 Introduction

By a **bar system** is meant any structure consisting of rod-shaped elements (see Fig. 1).

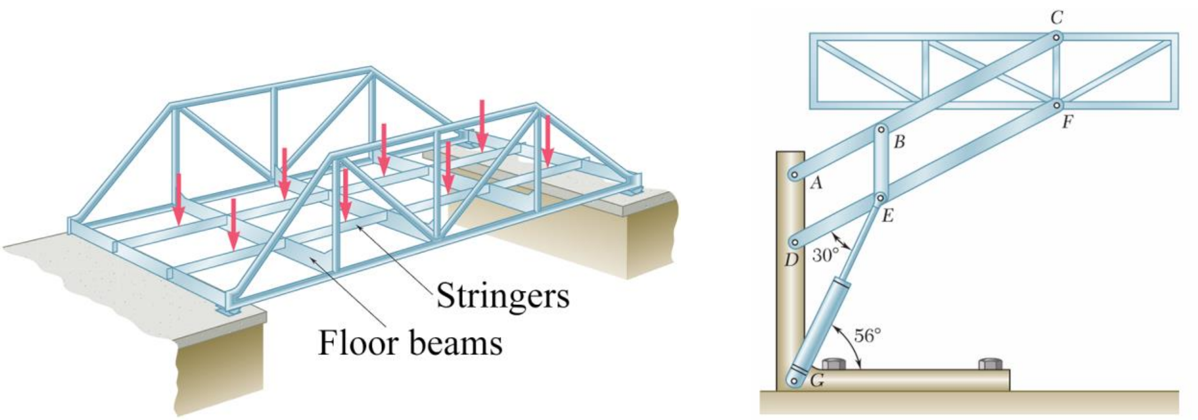


Fig. 1

If the elements of a bar system are primarily under tension or compression, the bar system is called a **truss** (see Fig. 2).

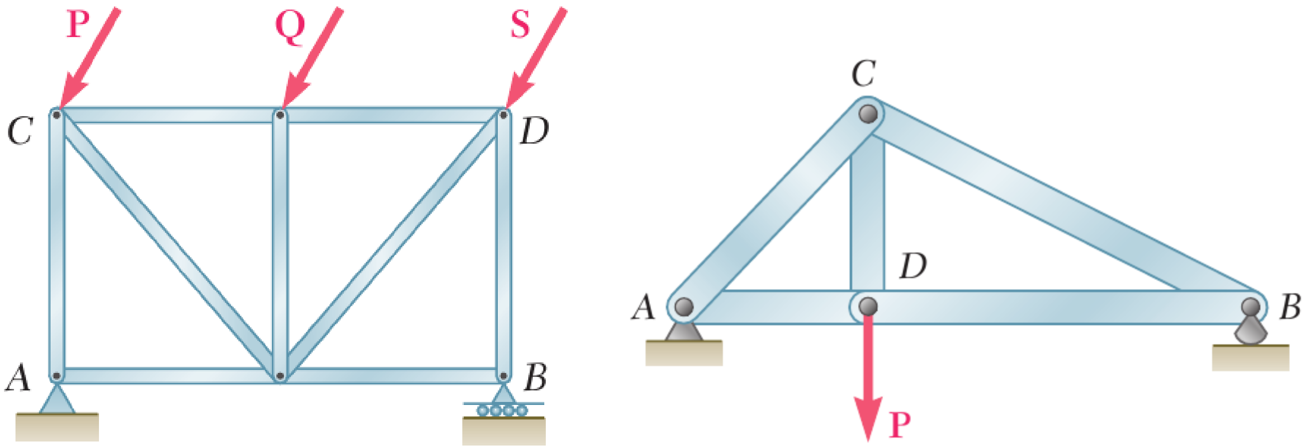


Fig. 2

If the elements of a bar system are primarily in bending or torsion the system is called a **frame** (see Fig. 3).

Frames are divided on statically determinate and statically indeterminate.

By a **statically determinate system** ***is meant a system for which all the reactions in the supports can be determined by means of equations of equilibrium and internal force factors at any cross section can also be found by the method of sections.***

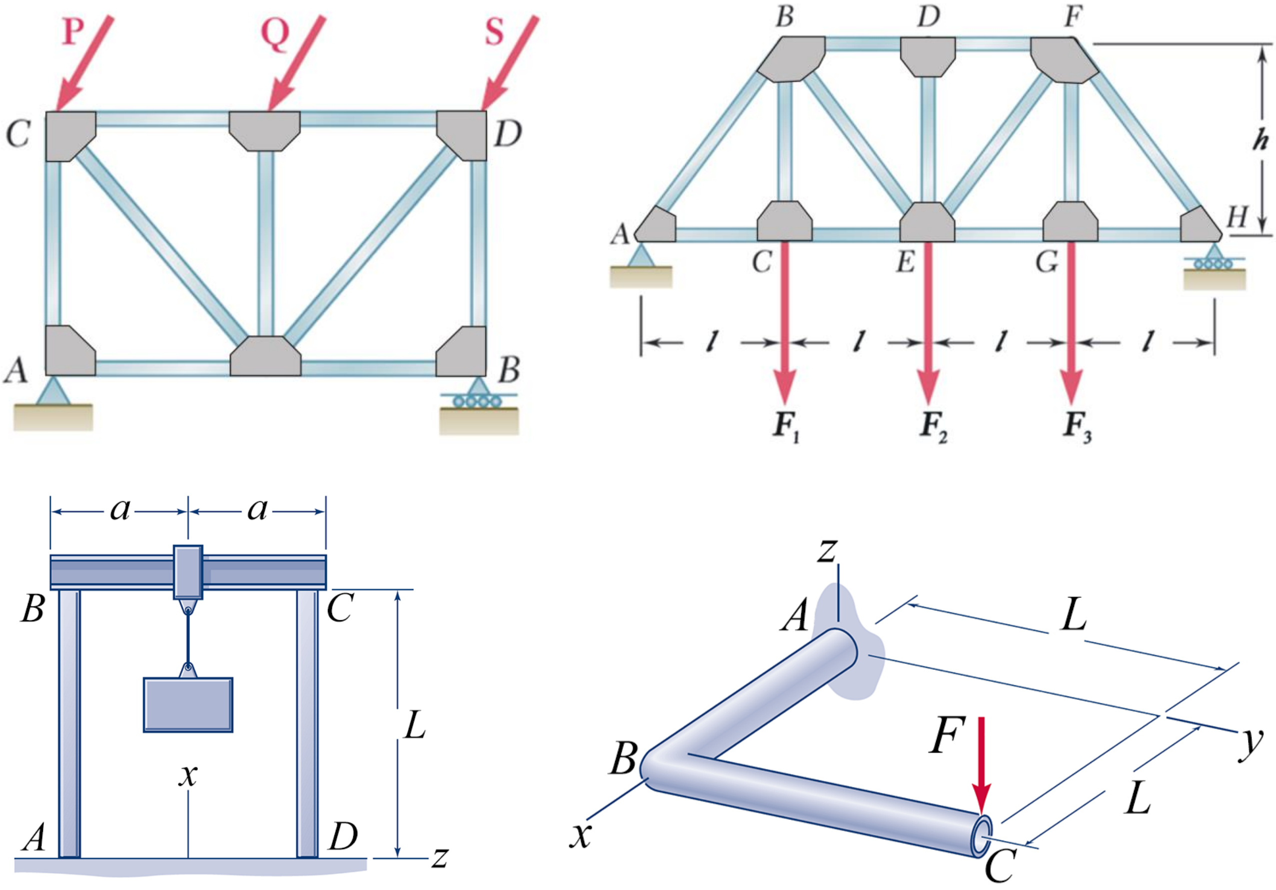


Fig. 3

By a **statically indeterminate system** ***is meant a system for which the external reactions and all internal force factors cannot be determined by means of the method of sections and equations of equilibrium*** (see Fig. 4)**.**

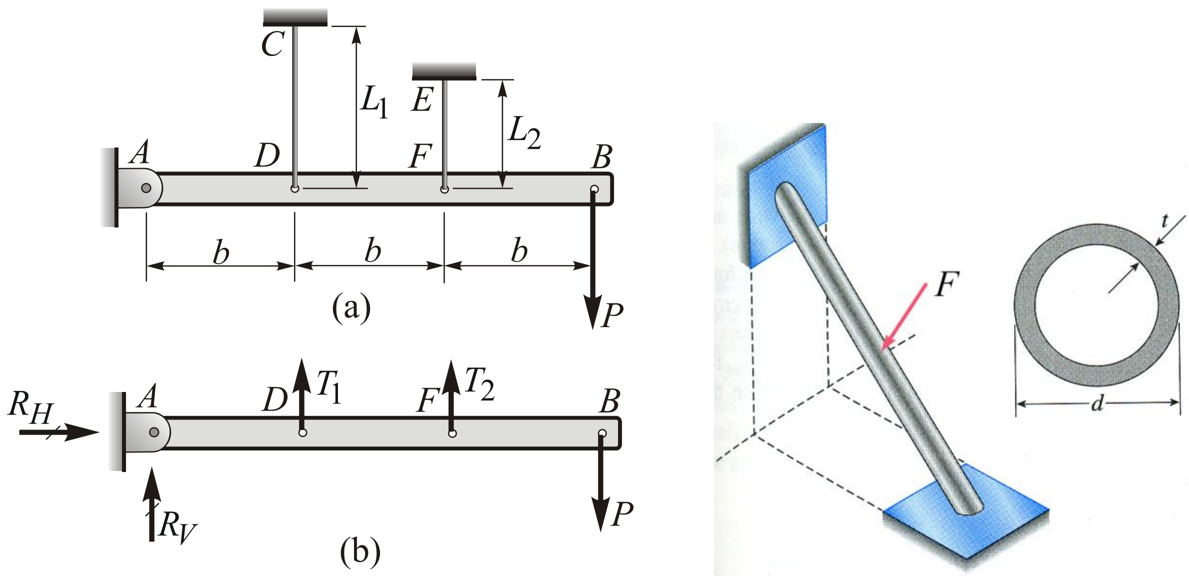


Fig. 4

The position of a rigid rod in space is determined by six independent coordinates. In other words, a rigid rod possesses six **degrees of freedom**. In equilibrium, any rod is subjected to **constraints**.

***The number of constraints required to attain kinematic invariability is termed the* required number of constraints**.

Any constraint imposed in excess of the required ones is called **redundant**. The number of redundant constraints is equal to the **degree of static indeterminacy** of the system.

***The difference between the number of unknown reactions of supports and the number of independent equations of statics that can be set up for a given system is called the* degree or number of static indeterminacy.** According to this number, systems are referred to as **singly**, **two-fold**, **three-fold**,… ***n*-fold statically indeterminate:**

$$K = m - n, \tag{1}$$

where K is the **degree of static indeterminacy**; m is an **actual number of constraints**; n is a **required number of constraints**.

Examples:

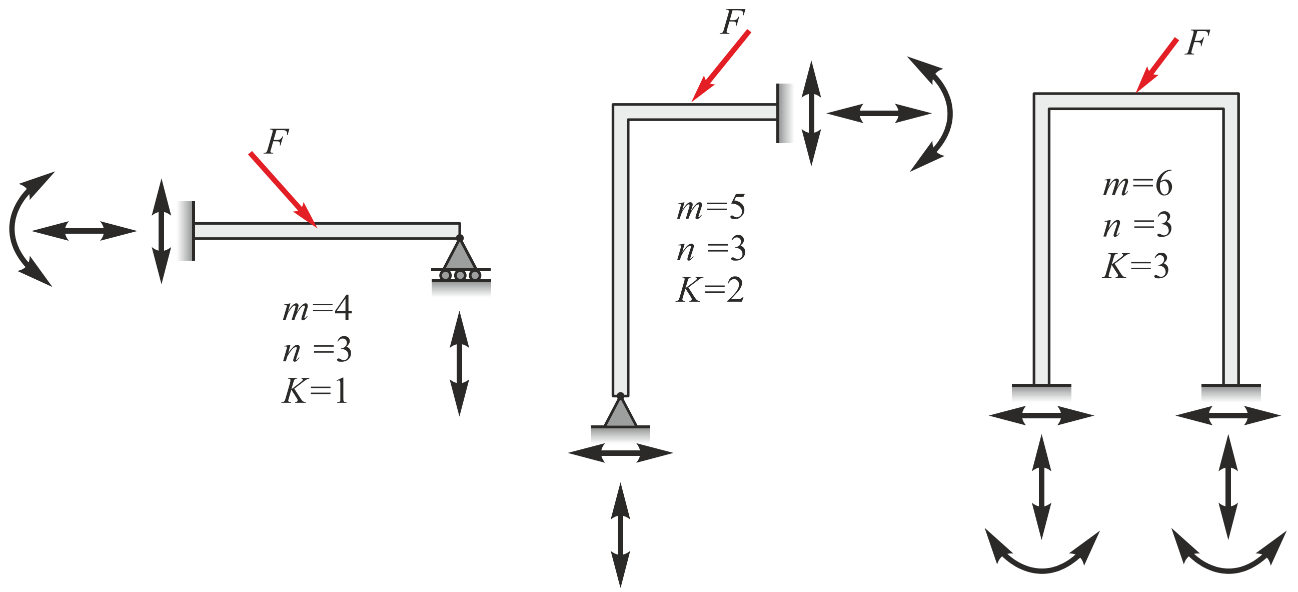


Fig. 5

(K is the number of static indeterminacy; m is the number of unknown reactions; n is the number of static equilibrium equations).

2 Essence of the Force Method

2.1 The analysis of any statically indeterminate frame by the force method begins with removing redundant constraints. ***A system relieved from redundant constraints is termed a* base system**. Base system is statically determinate.

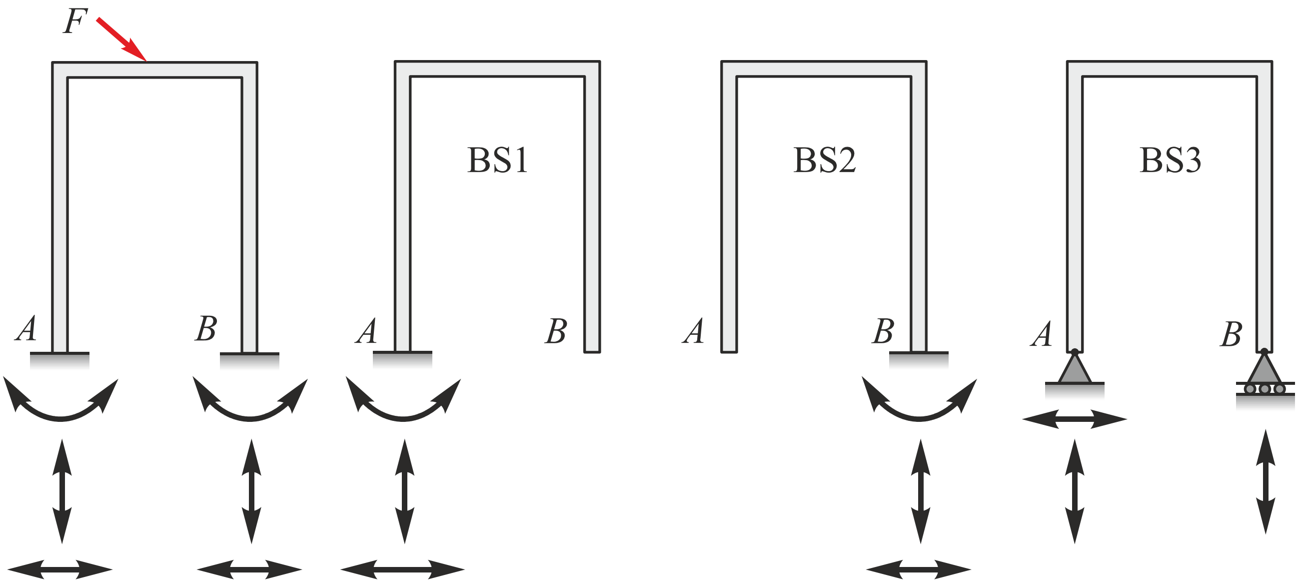


Fig. 6

Note, that base system can not be kinematically changeable.

2.2. After this, it is necessary to introduce unknown force factors in the points of redundant constraints. Note, that forces are introduced at the sections where linear displacements are prevented. And moments are introduced at the sections where angular displacements are prevented**.** In both cases, unknown force factors are denoted by X_1, X_2, …, X_i, where i is the number of the unknown force.

***Base system with unknown reactions of redundant constraints and external force factors is termed* equivalent system.**

Example of equivalent system (for base system BS1) is shown on Fig. 7.

2.3 Proceed now to derivation of equations for unknown force factors determining (Fig. 7). In our case the horizontal, vertical and angular displacements at the point B are equal to zero:

$$f_{horiz_{p.B}} = \Delta_1 = 0,\quad f_{vert_{p.B}} = \Delta_2 = 0,\quad f_{angul_{p.B}} = \Delta_3 = 0. \tag{2}$$

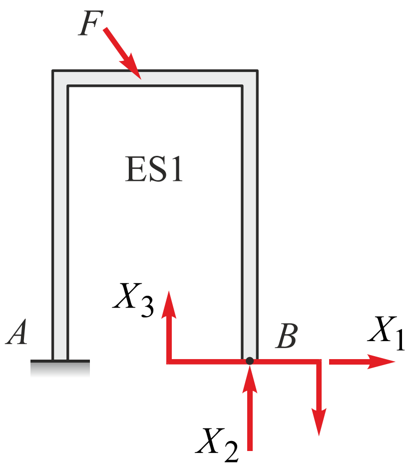
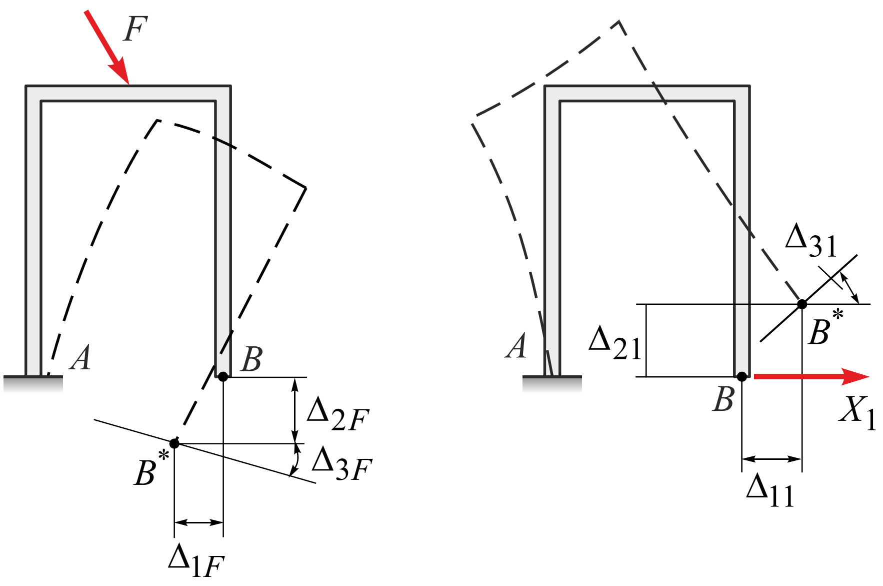


Fig. 7

By using the principle of superposition, we rewrite the expressions for the displacements

$$\left.\begin{aligned} \Delta_1 &= \Delta_{11} + \Delta_{12} + \Delta_{13} + \Delta_{1F} = 0, \\ \Delta_2 &= \Delta_{21} + \Delta_{22} + \Delta_{23} + \Delta_{2F} = 0, \\ \Delta_3 &= \Delta_{31} + \Delta_{32} + \Delta_{33} + \Delta_{3F} = 0, \end{aligned}\right\} \tag{3}$$

where Δ_{ik} is the displacement of the point "i" in the direction of the force (reaction) "i" caused by the reaction "k"; Δ_{iF} is the displacement of the point i in the direction of the reaction i caused by the external forces (Fig. 8)



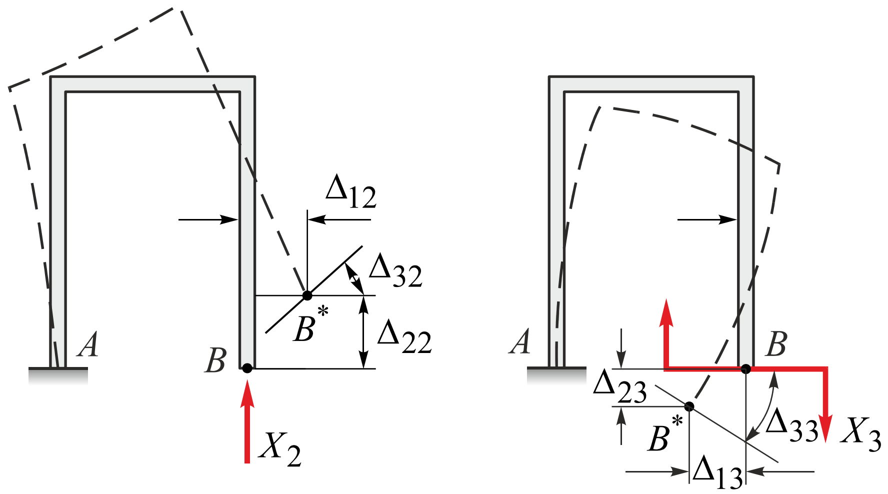


Fig. 8

Since each displacement is proportional to the corresponding force the quantity Δ_{ik} may be written as

$$\Delta_{ik} = \delta_{ik} \times X_k, \quad (4)$$

where δ_{ik} is the displacement in the direction of the *i*-th force factor (force or moment) produced by the **unit** factor replacing the *k*-th factor. We substitute the expression Δ_{ik} into system (3):

$$\begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \Delta_{1F} = 0, \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \Delta_{2F} = 0, \\ \delta_{31}X_1 + \delta_{32}X_2 + \delta_{33}X_3 + \Delta_{3F} = 0. \end{cases} \quad (5)$$

– system of canonical equations for 3 - fold statically indeterminate rod system.

For *k*-degree of static indeterminacy, system of canonical equations is the following:

$$\left.\begin{array}{l} \delta_{11}X_1 + \delta_{12}X_2 + \ldots + \delta_{1k}X_k + \Delta_{1F} = 0, \\ \delta_{21}X_1 + \delta_{22}X_2 + \ldots + \delta_{2k}X_k + \Delta_{2F} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \delta_{k1}X_1 + \delta_{k2}X_2 + \ldots + \delta_{kk}X_k + \Delta_{kF} = 0. \end{array}\right\} \quad (6)$$

These equations are final and are known as the **canonical equations of the force method**.

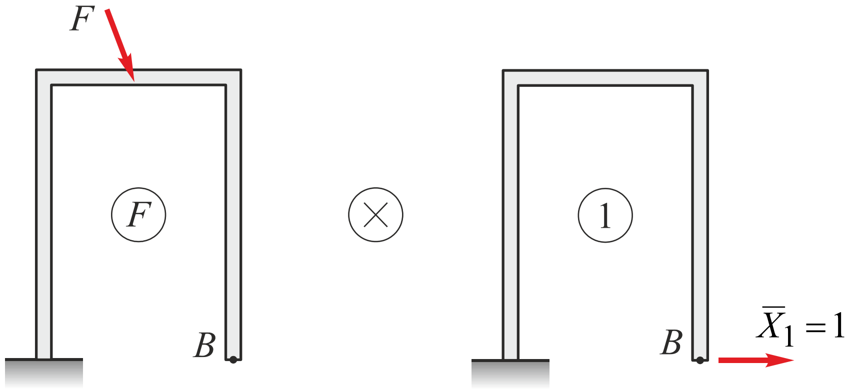
According to the reciprocal theorem

(a) $\delta_{ik} = \delta_{ki}$, (7)

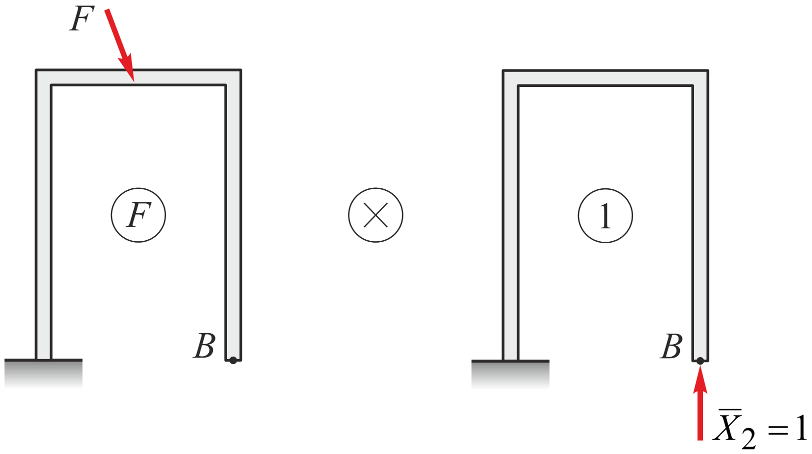
(b) $\delta_{ik} > 0$ with $i=k$. (8)

Since the coefficients δ_{11}, δ_{12}, δ_{21}, δ_{22}, δ_{23}, δ_{32}, δ_{33}, δ_{31}, δ_{13}, Δ_{1F}, Δ_{2F}, Δ_{3F} are really the generalized displacements, they are calculated applying Mohr's method or its graphical derivative. Corresponding force (F) and unit (1) systems are designed for this purpose:

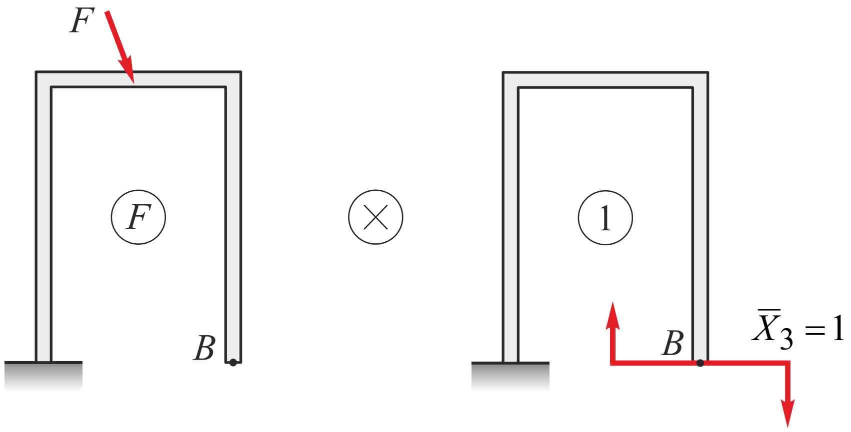
(a) Δ_{1F} calculating:



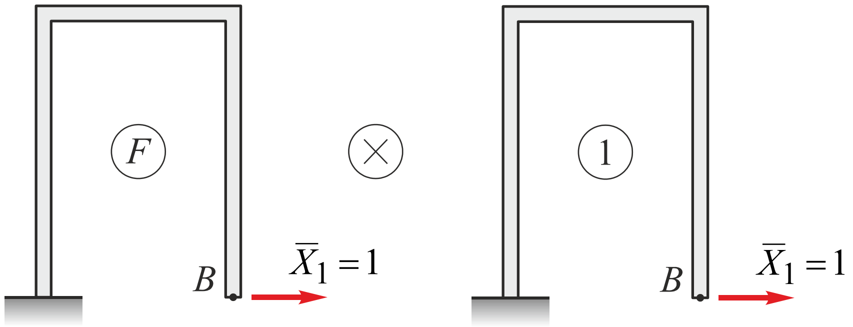
(b) Δ_{2F} calculating:



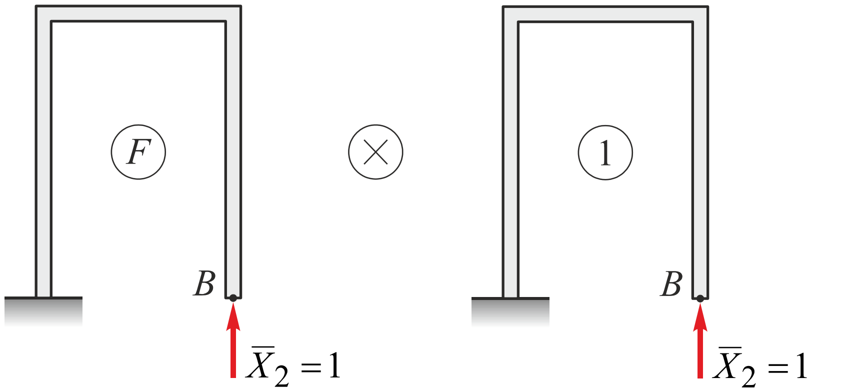
(c) Δ_{3F} calculating:



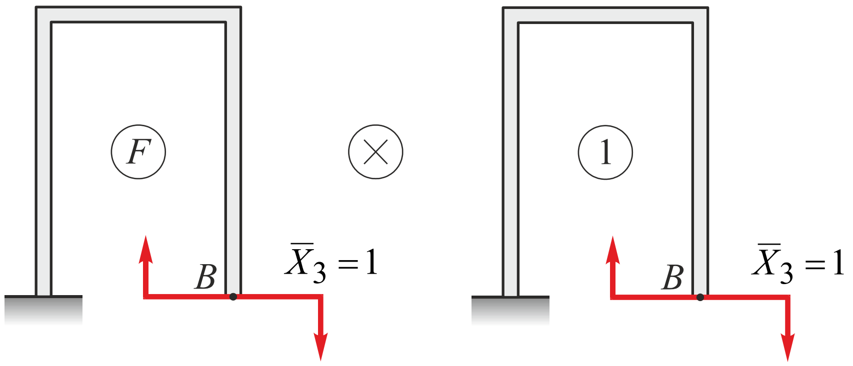
(d) δ_{11} calculating:



(e) δ_{22} calculating:

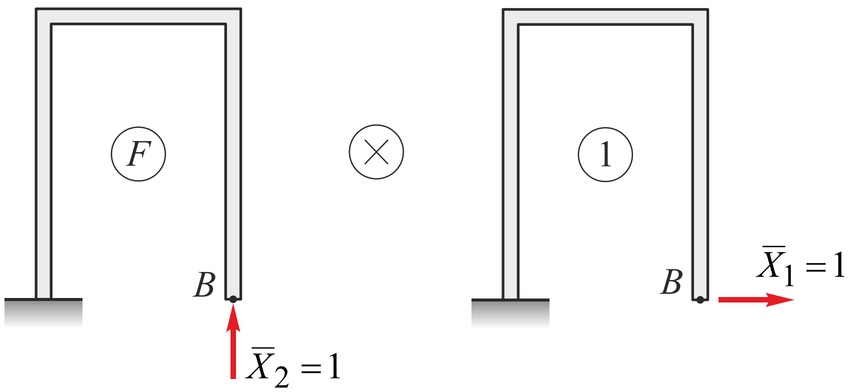


(f) δ_{33} calculating:



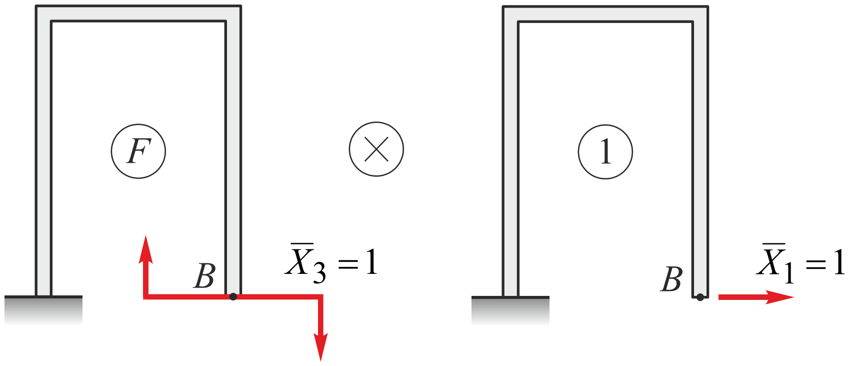
(g) $\delta_{12} = \delta_{21}$

calculating:



(h) $\delta_{13} = \delta_{31}$

calculating:



(i) $\delta_{23} = \delta_{32}$

calculating:

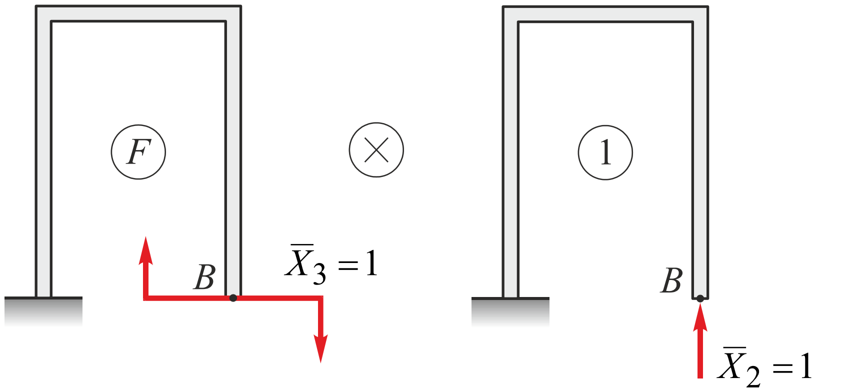


Fig. 9

Analyzing the frames on Fig. 9, we should conclude that the number of different in essence rod systems used for δ_{ik} calculating may be decreased to the following shown on Fig. 10 for 3-fold statically indeterminate frame.

By substituting δ_{ik} into canonical equations (5) we will find X_1, X_2, X_3 and consider statically determinate equivalent system (Fig. 7) to calculate internal forces in its cross-sections.

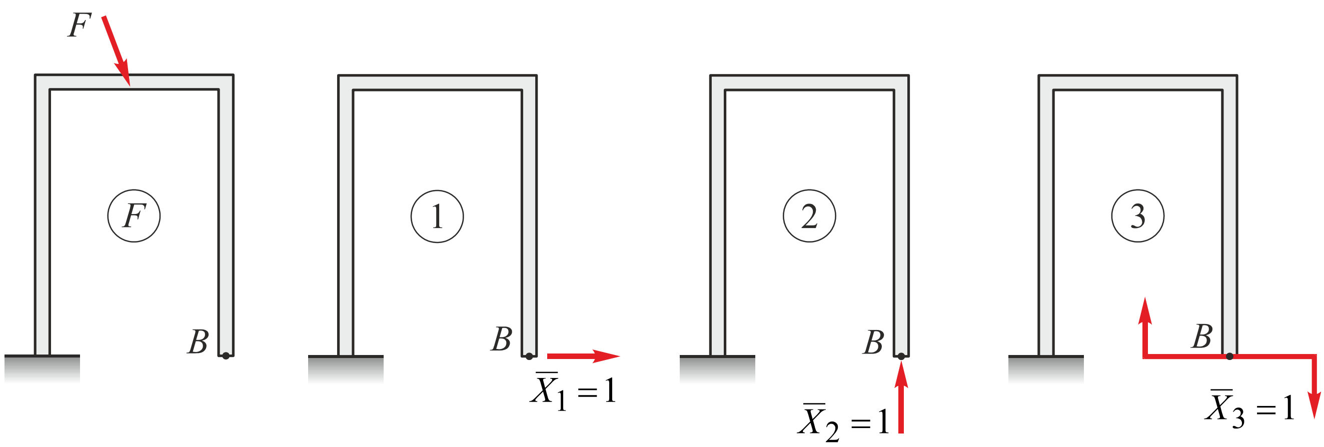


Fig. 10