LECTURE 23 Examples to Lecture 22

Example 1 Singly statically indeterminate frame

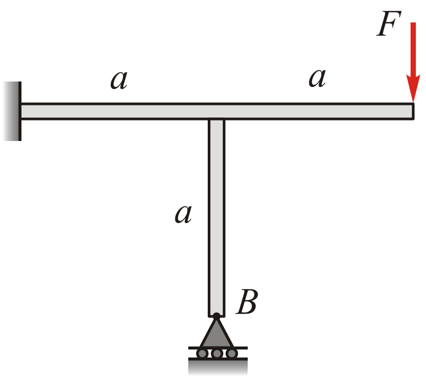


Fig. 1

Given: F, a, EI_y = const

R. D.: Open static indeterminacy and draw the normal force, shearing force and bending moment diagrams for the frame shown in Fig. 1.

1. Determining the degree of static indeterminacy:

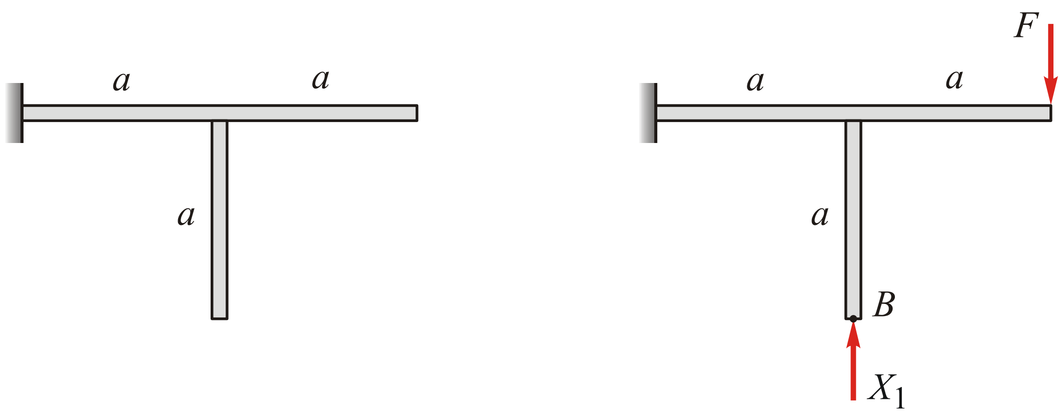
$m = 4$,

$n = 3$,

$K = 1$.

The **frame is singly statically indeterminate**.

2. We choose a base system by removing the right support (Fig.2):



Base system (BS) *Equivalent system (ES)*

Fig. 2

3. Draw the equivalent system (Fig. 2). The effect of the support is replaced by the force X_1.

The canonical equation for the system under consideration become

$$\delta_{vert\ p.B} = \delta_{11} X_1 + \Delta_{1F} = 0.$$

The primary displacements in the frame are induced by bending. Hence, neglecting the shear and tension-compression of the bars, we draw the bending moment diagrams due to the given force F and due to unit force $\bar{X}_1$:

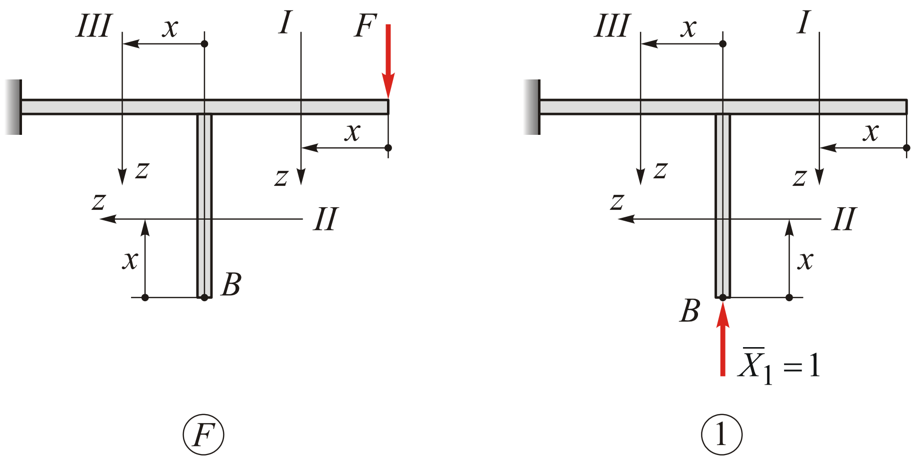


Fig. 3

Compile a table of bending moments according to the given portions:

Table 1

No	Limits of portion	M_{y_F}	$\bar{M}_{y_1}$
I-I	$0 \le x \le a$	$-Fx$	0
II-II	$0 \le x \le a$	0	0
III-III	$0 \le x \le a$	$-F(a+x) = -Fx - Fa$	$1 \cdot x = x$

Corresponding graphs of bending moments are:

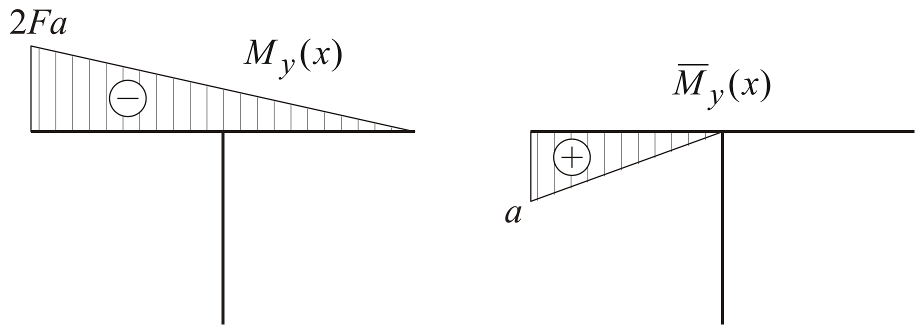


Fig. 4

We determine the coefficients of the canonical equation assuming that the rigidity of all portions of the frame is constant and equals to EI_y.

The quantity δ_{11} is determined by multiplying the unit diagram by itself. Hence the area under the diagram for each portion is multiplied by the ordinate of the same diagram passing through its centroid:

$$\delta_{11} = \left[\left(\frac{a \times a}{2} \right) \left(\frac{2}{3} a \right) \right] \frac{1}{EI_y} = \frac{a^3}{3EI_y}.$$

We note that ***the quantities*** δ_{ik} ***with*** $i=k$ ***are always positive*** since the areas under the diagrams and the ordinates are of the same sign.

Further we determine the next coefficient Δ_{1F} in the equation by the use of Mohr's method:

$$\Delta_{1F} = \frac{1}{EI_y}\left[\int_0^a (-Fx)(0)dx + \int_0^a (0)(0)dx + \int_0^a [(-F)(x+a)]xdx\right] = -\frac{5}{6}\frac{Fa^3}{EI_y}.$$

We substitute the coefficients so derived in the canonical equation.

$$\frac{1}{3}\frac{a^3}{EI_y}X_1 - \frac{5}{6}\frac{Fa^3}{EI_y} = 0, \qquad X_1 = +2,5F.$$

This completes the opening of static indeterminacy.

It becomes possible to draw an internal force factors diagrams for the equivalent system:

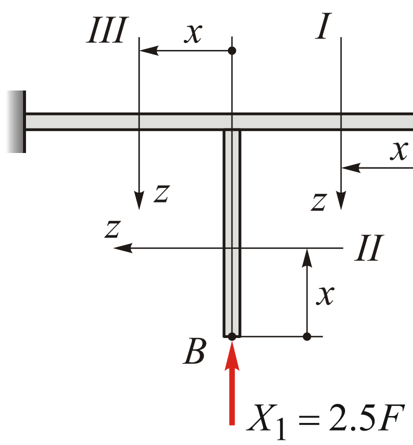


Fig. 5

Note that after calculation of X_1 equivalent system becomes statically determinate.

Let us draw an internal forces diagrams:

I-I $\quad 0 \le x \le a \qquad N_y^I(x) = 0, \qquad Q_z^I(x) = +F,$

$$M_y^I(x) = -Fx\big|_{x=0} = 0\big|_{x=a} = -Fa.$$

II-II $\quad 0 \le x \le a \qquad N_y^{II}(x) = -X_1 = -2.5F, \quad Q_z^{II}(x) = 0, \quad M_y^{II}(x) = 0.$

III-III $0 \le x \le a$ $\quad N_y^{III}(x) = 0, \quad Q_z^{III}(x) = F - X_1 = F - 2.5F = -1.5F,$

$$M_y^{III}(x) = -F(x+a) + X_1 a \Big|_{x=0} = -Fa \Big|_{x=a} = \frac{1}{2} Fa .$$

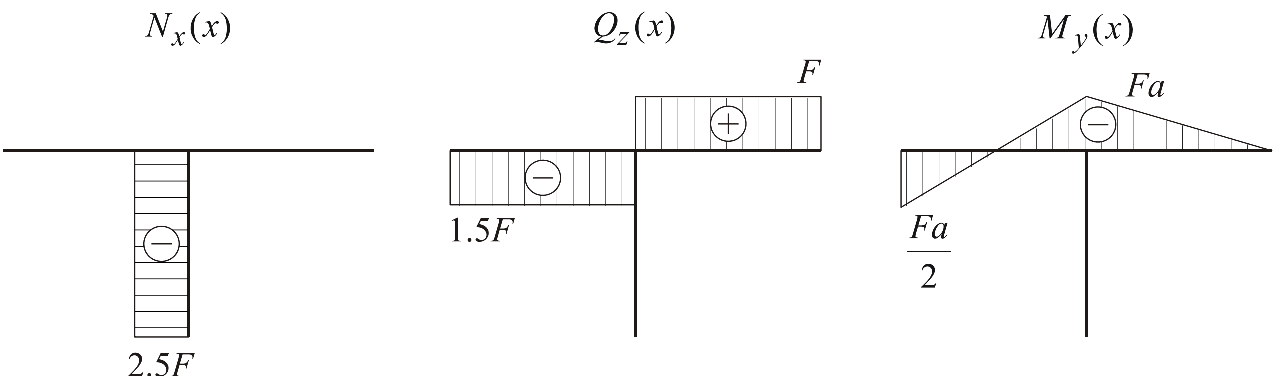


Fig. 6

Checking the accuracy of result, i.e. checking the equilibrium:

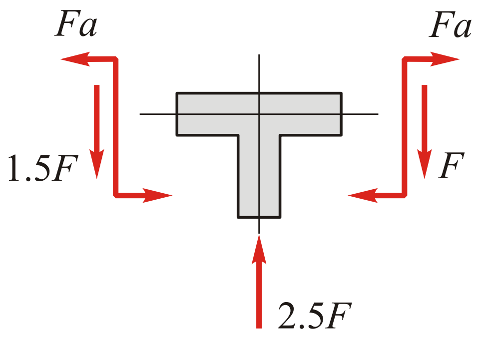


Fig. 7

Example 2 Two-fold statically indeterminate frame

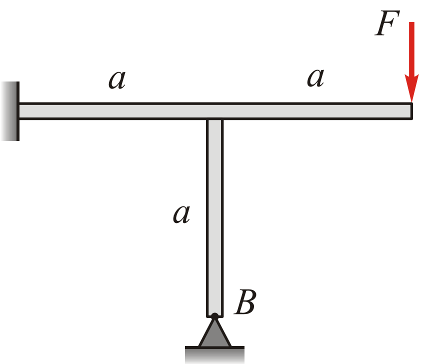


Fig. 1

Given: F, a, EI_y = const

R. D.: Open static indeterminacy and draw the normal force, shearing force and bending moment diagrams for the frame shown in Fig. 1

1. Determine the degree of static indeterminacy

$m = 5,$

$n = 3,$

$K = 2.$

The **frame is two-fold statically indeterminate.**

2. We choose a base system by removing the right support (Fig. 2):

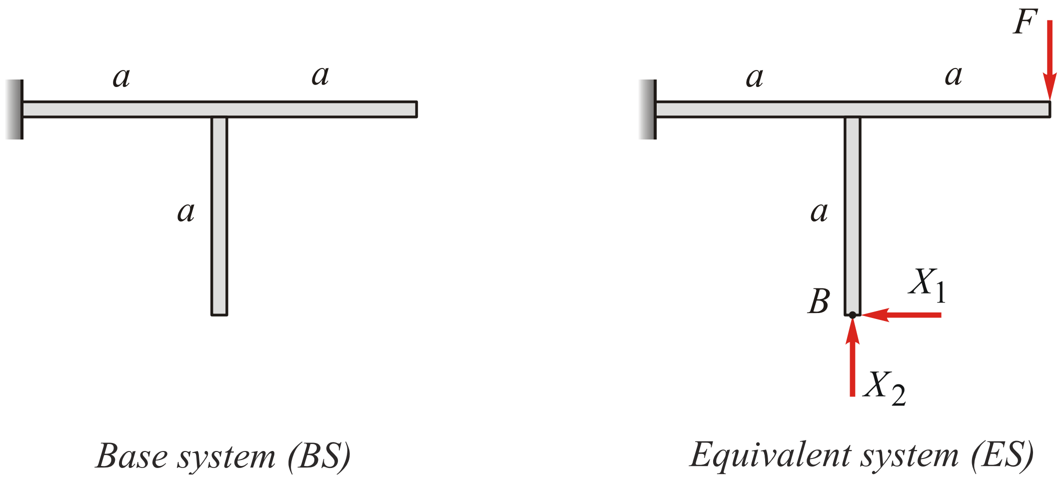


Fig. 2

3. Draw the equivalent system (Fig. 2). The effect of the support is replaced by two forces X_1 and X_2 .

The canonical equations for the system under consideration become

$$\begin{cases} \delta_{hor\ p.B} = \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1F} = 0, \\ \delta_{vert\ p.B} = \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2F} = 0. \end{cases}$$

The primary displacements in the frame are induced by bending. Hence, neglecting the shear and tension-compression of the bars, we draw the bending moment diagrams due to the given force F and due to two unit force factors:

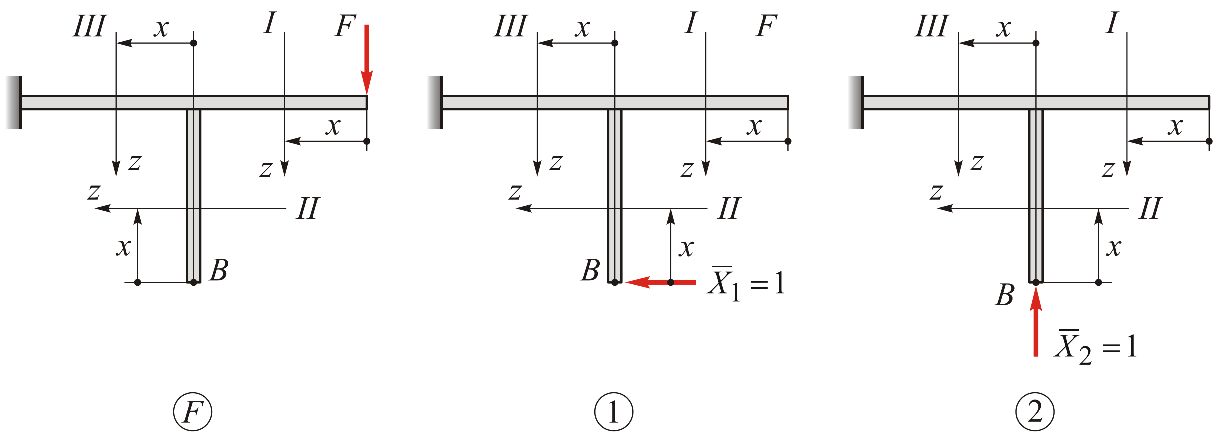


Fig. 3

Compile a Table 2 of bending moments according to the given portions:

Table 2

No	Limits of portion	M_{y_F}	$\bar{M}_{y_1}$	$\bar{M}_{y_2}$
I-I	$0 \le x \le a$	$-Fx$	0	0
II-II	$0 \le x \le a$	0	$-1 \times x = -x$	0
III-III	$0 \le x \le a$	$-F(a+x) = -Fx - Fa$	$-1 \times a = -a$	$1 \times x = x$

Corresponding graphs of bending moments are:

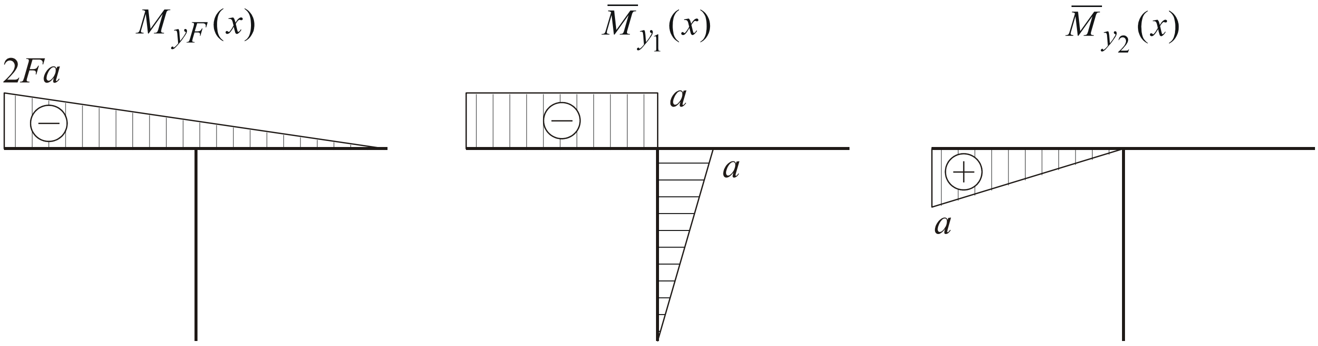


Fig. 4

We determine the coefficients of the canonical equations assuming that the rigidity of all portions of the frame is constant and equals to EI_y.

The quantity δ_{11} is determined by multiplying the first unit diagram by itself. Hence the area under the diagram for each portion is multiplied by the ordinate of the same diagram passing through its centroid:

$$\delta_{11} = \left[\left(-\frac{a \times a}{2}\right)\left(-\frac{2}{3}a\right) + (-a \times a)(-a)\right]\frac{1}{EI_y} = \frac{4}{3}\frac{a^3}{EI_y}.$$

We note that quantities δ_{ik} with $i = k$ are always positive since the areas under the diagrams and the ordinates are of the same sign.

Further we determine the coefficients in the equations by multiplying the diagrams with the appropriate numbers:

$$\delta_{22} = \frac{1}{EI_y}\left[\left(\frac{a \times a}{2}\right)\left(\frac{2}{3}a\right)\right] = \frac{1}{3}\frac{a^3}{EI_y},$$

$$\delta_{21}=\delta_{12}=\frac{1}{EI_y}\left[(0\times 0)+(0)\left(-\frac{a\times a}{2}\right)+\left(\frac{a\times a}{2}\right)(-a)\right]=-\frac{1}{2}\frac{a^3}{EI_y}.$$

By the use of Mohr's method we determine the remaining coefficients Δ_{1F} and Δ_{2F} in the equations:

$$\Delta_{1F}=\frac{1}{EI_y}\left[\int_0^a -Fx(0)dx+\int_0^a(0)(-x)dx+\int_0^a(-Fx-Fa)(-a)dx\right]=\frac{3}{2}\frac{Fa^3}{EI_y},$$

$$\Delta_{2F}=\frac{1}{EI_y}\left[\int_0^a(-Fx-Fa)(x)dx\right]=\frac{1}{EI}\left[-\frac{Fx^3}{3}-Fa\frac{x^2}{2}\right]\Bigg|_0^a=-\frac{5}{6}\frac{Fa^3}{EI_y}.$$

Next step is to substitute derived coefficients in the canonical equations. This completes the opening of static indeterminacy:

$$\begin{matrix}\frac{4}{3}a^3X_1-\frac{a^3}{2}X_2+\frac{3}{2}Fa^3=0,\\ -\frac{a^3}{2}X_1+\frac{a^3}{3}X_2-\frac{5}{6}Fa^3=0,\end{matrix}\quad \text{or}\quad \begin{cases}\frac{4}{3}X_1-\frac{1}{2}X_2+\frac{3}{2}F=0,\\ -\frac{1}{2}X_1+\frac{1}{3}X_2-\frac{5}{6}F=0.\end{cases}$$

After solution, $X_1=-\frac{3}{7}F$, $X_2=\frac{13}{7}F$.

Note. Negative sign of X_1 means that its actual direction is opposite to original.

After this we draw the internal force factors diagrams for the equivalent system which becomes statically determinate:

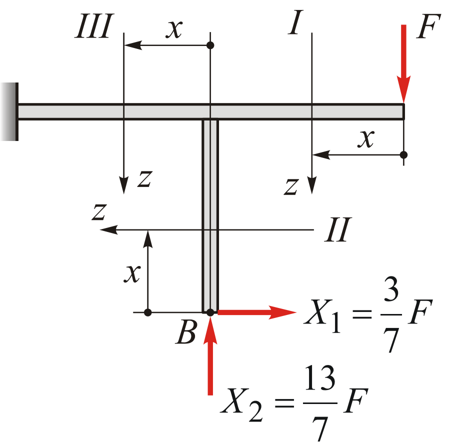


Fig. 5

I-I: $0\le x\le a$ $N_y^I(x)=0$, $Q_z^I(x)=+F$,

$$M_y^I(x)=-Fx\big|_{x=0}=0\big|_{x=a}=-Fa. \qquad (*)$$

II-II: $0\le x\le a$ $N_y^{II}(x)=-X_2=-\frac{13}{7}F$,

$$Q_z^{II}(x)=-X_1=-\frac{3}{7}F,$$

$$M_y^{II}(x)=X_1x\big|_{x=0}=0\big|_{x=a}=\frac{3}{7}Fa.$$

III-III: $0 \le x \le a \quad N_y^{III}(x) = +X_1 = +\frac{3}{7}F,$

$$Q_z^{III}(x) = F - X_2 = F - \frac{13}{7}F = -\frac{6}{7}F,$$

$$M_y^{III}(x) = -F(x+a) + X_1 a + X_2 x\Big|_{x=0} = -Fa + \frac{3}{7}Fa = -\frac{4}{7}Fa\Big|_{x=a} = \frac{2}{7}Fa.$$

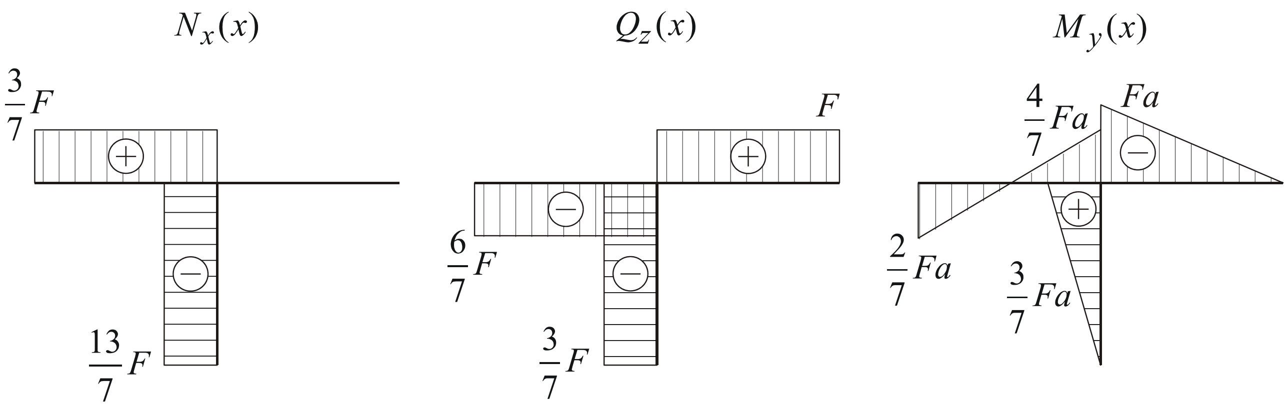


Fig. 6

Checking the accuracy of result

1. Checking the equilibrium:

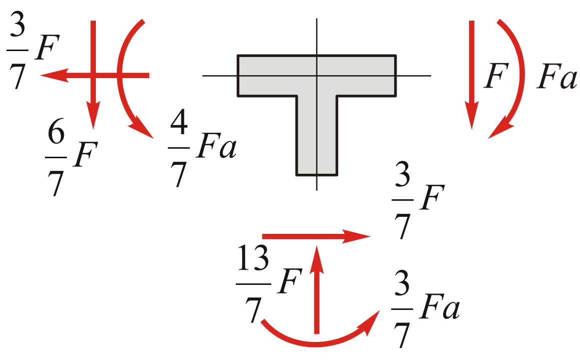


Fig. 7

2. Determining the evidently zero displacement, for example, the slope in point *A*.

For this, ***we choose new base system by removing the redundant left rigid support***:

Designing the new equivalent system:

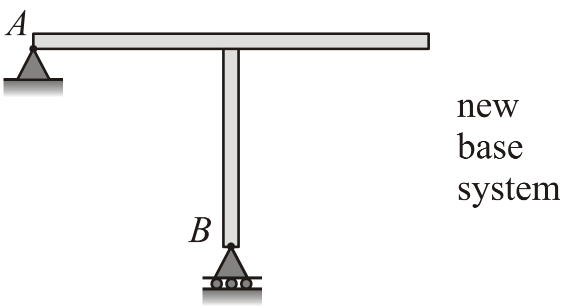


Fig. 8

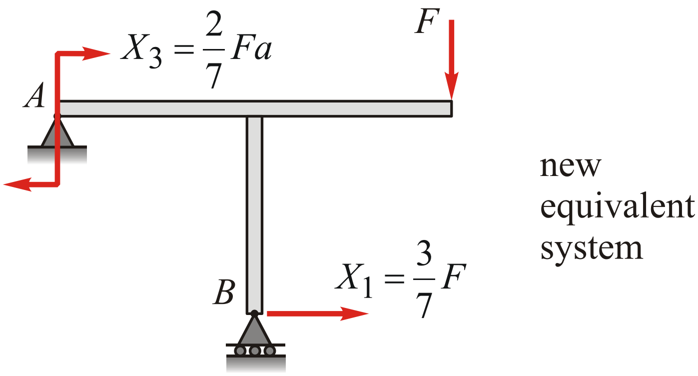


Fig. 9

It is evident, that internal forces in new equivalent system are described by the equations (*). To calculate evidently zero slope on the left support of new equivalent system, it is necessary to consider it as the **force system** and also consider the next **unit system**:

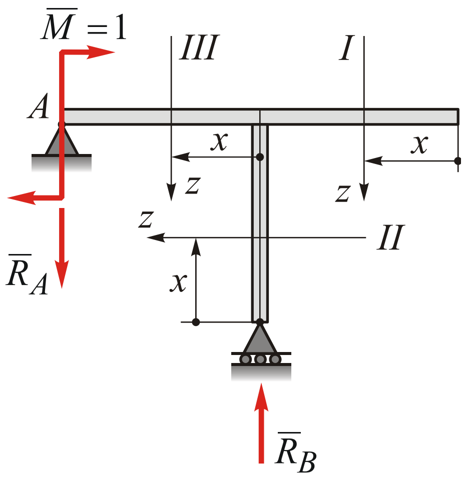


Fig. 10

Internal forces in new unit system are described by the equations:

$$\bar{R}_A = \frac{1}{a}, \quad \bar{R}_B = \frac{1}{a}.$$

$$\bar{M}_y^I(x) = 0; \quad \bar{M}_y^{II}(x) = 0; \quad \bar{M}_y^{III}(x) = +\frac{1}{a}x.$$

The slope in point A is calculated by the following Mohr's integral:

$$\theta_A = \frac{1}{EI_y}\left[\int_0^a M_y^I(x)\bar{M}_y^I(x)dx + \int_0^a M_y^{II}(x)\bar{M}_y^{II}(x)dx + \int_0^a M_y^{III}(x)\bar{M}_y^{III}(x)dx\right] =$$

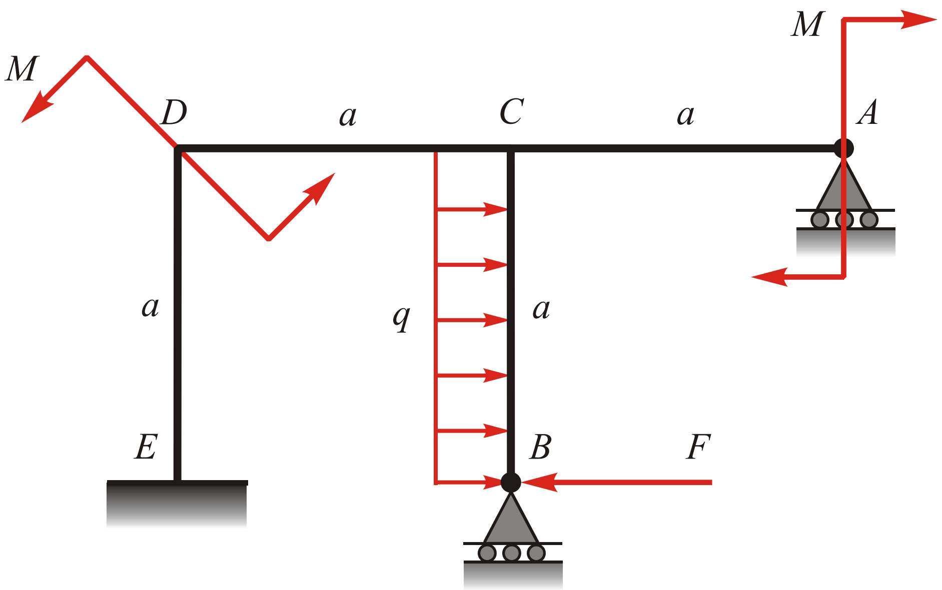
$$=\frac{1}{EI_y}\left[\int_0^a(-Fx)\times(0)\times dx+\int_0^1\left(\frac{3}{7}Fx\right)(0)\,dx+\int_0^a\left(+\frac{6}{7}Fx-\frac{4}{7}Fa\right)\left(\frac{x}{a}\right)dx\right]=$$

$$=\frac{1}{EI_y}\left[\left(\frac{6}{7}\frac{F}{a}\right)\left(\frac{x^3}{3}\right)\Bigg|_0^a-\left(\frac{4}{7}F\right)\left(\frac{x^2}{2}\right)\Bigg|_0^a\right]=\frac{1}{EI_y}\left[\frac{2}{7}Fa^2-\frac{2}{7}Fa^2\right]=0\,.$$

Example 3 Two-fold statically indeterminate frame

Data: q = 10 kN/m, F= 20 kN, M = 10 kNm, a = 2 m.

Goal: 1) Open static indeterminacy, 2) Draw the graphs $N_x(x)$, $Q_z(x)$, $M_y(x)$.



Solution

(1) Degree of static indeterminacy

$K=m-n$, where $m=5$ – total number of constraints,

$n=3$ – minimum number of constraints.

After substituting, $K=5-3=2$.

Conclusion: plane frame is 2-fold statically indeterminate.

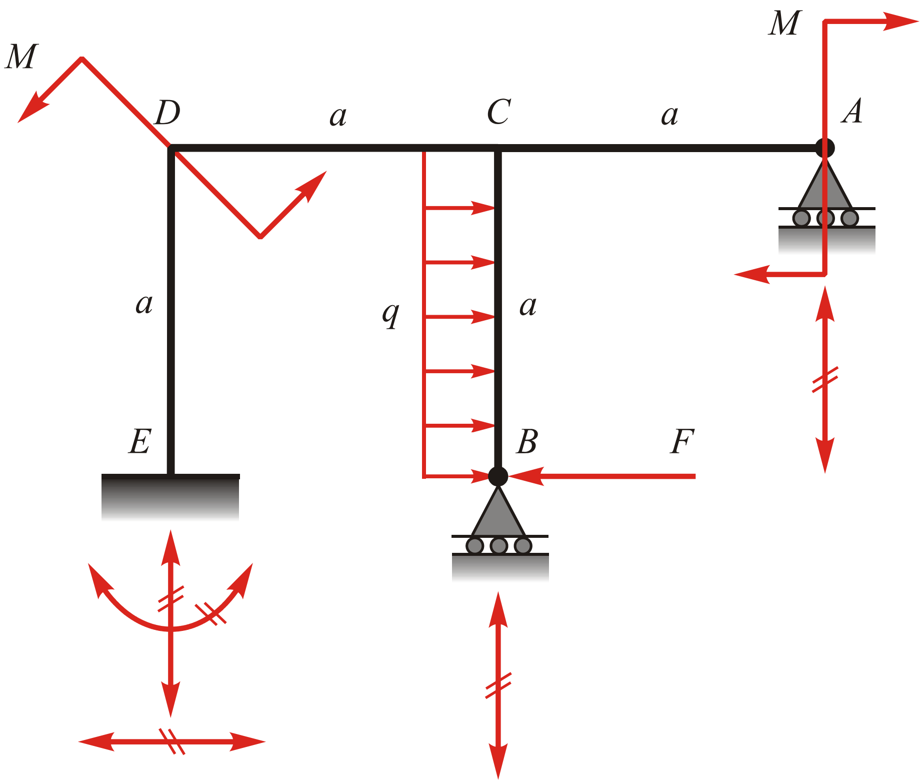


Fig. 1 Plane frame in equilibrium under external loading and reactions of supports

(2) Selecting the one of base systems

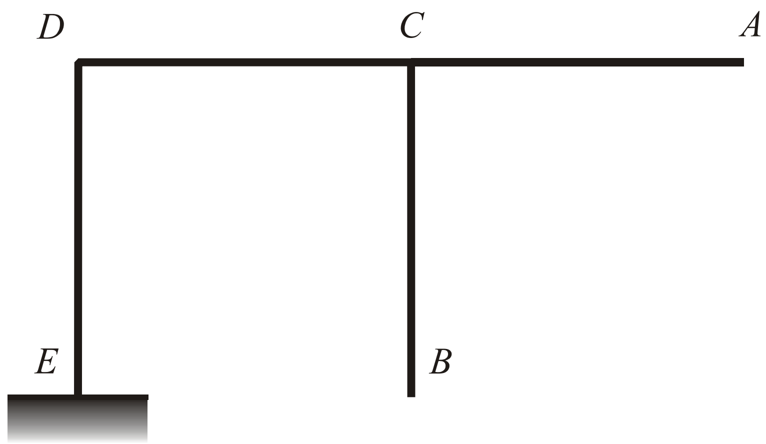


Fig. 2 Selected base system

Note. ***Base system should be statically determinate*****.**

(3) Designing the equivalent system

Note. ***To design the equivalent system, it is necessary to impose on the base system external forces and also the reactions of redundant constrains*** X_1 ***and*** X_2**.**

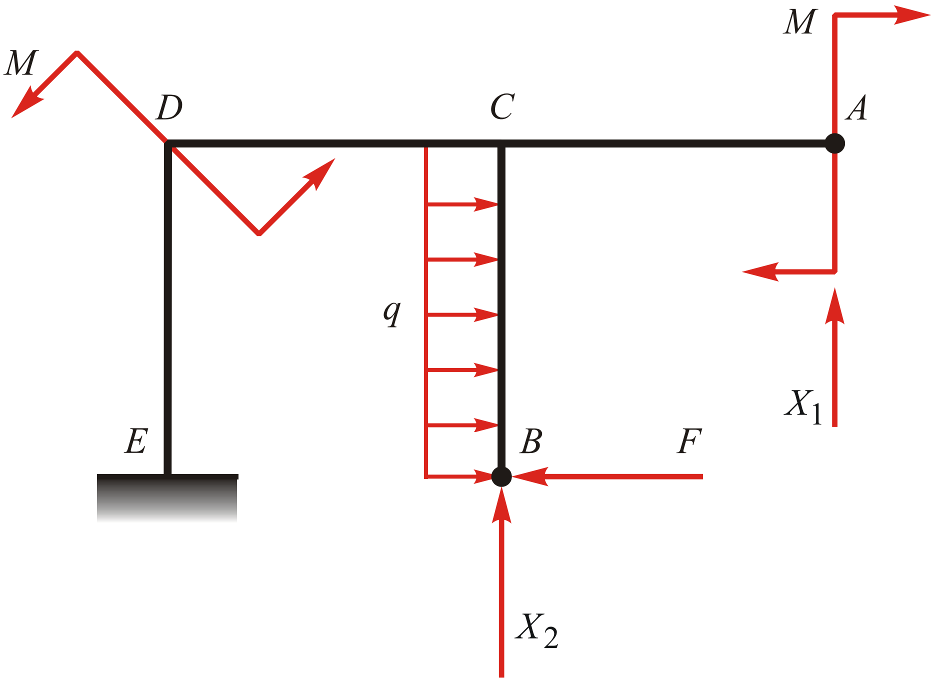


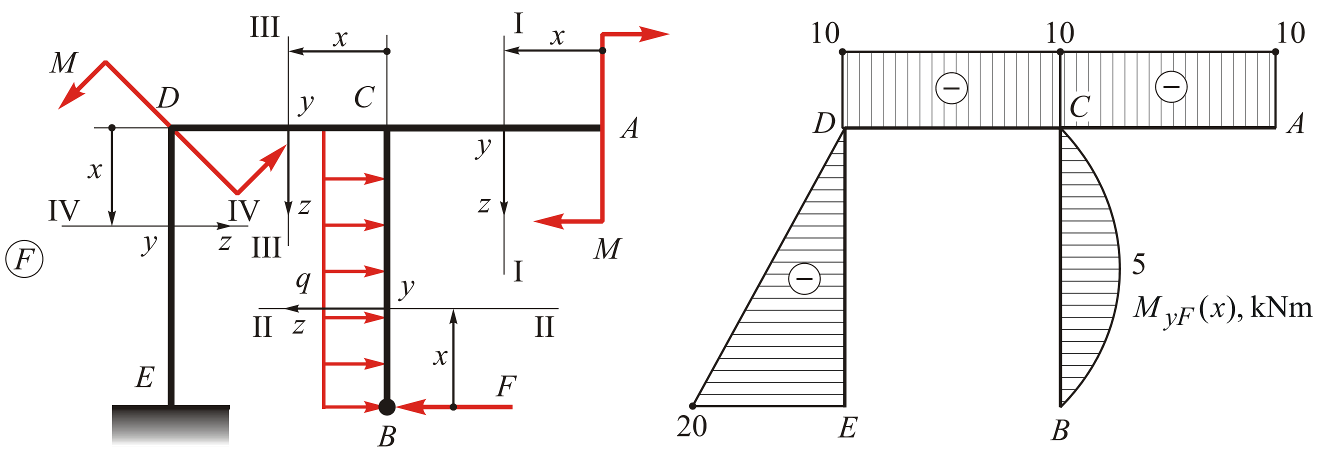
Fig. 3 Designed equivalent system

(4) Writing the system of canonical equations (compatibility equations) taking into consideration evidently zero vertical displacements of A and B points in given system.

$$\begin{cases}\delta_{vert\cdot A}(X_1, X_2, F) = 0,\\ \delta_{vert\cdot B}(X_1, X_2, F) = 0,\end{cases} \quad \text{or in canonical shape} \quad \begin{cases}\delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1F} = 0,\\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2F} = 0.\end{cases}$$

(5) Calculating the coefficients of canonical equations.

To find six coefficients δ_{11}, δ_{12}, δ_{21}, δ_{22}, Δ_{1F}, Δ_{2F}, it is necessary to consider the force system (F) and two unit systems: (1) and (2). These systems are shown on Fig 4.



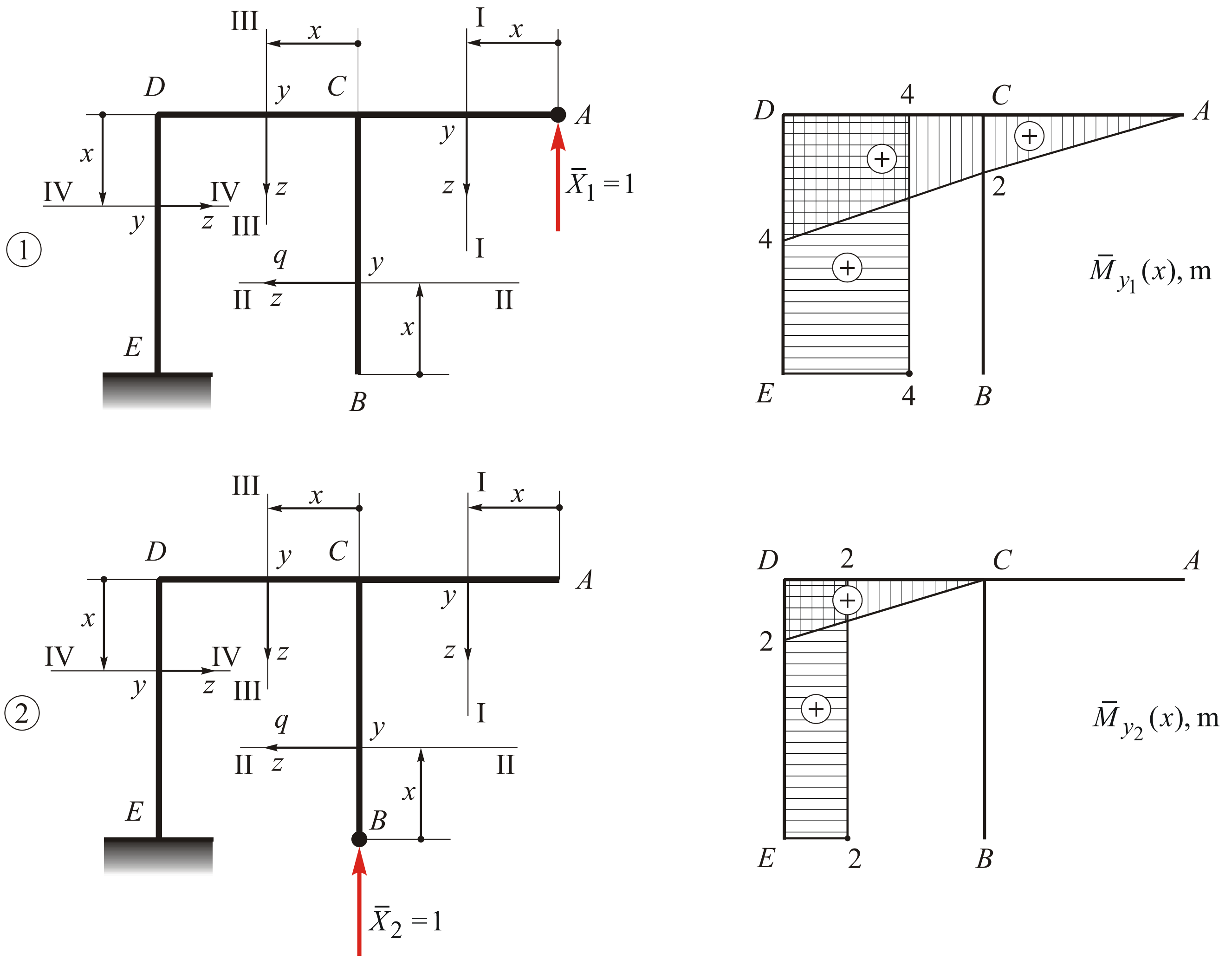


Fig. 4 The force system (F) (above left) and two unit systems with corresponding graphs of bending moments

By applying the method of sections the equations of internal forces are

Portion I – I $(0 < x < 2\ \text{m})$

$$M^{I}_{yF}(x) = -M = -10\ \text{kNm},$$

$$\overline{M}^{I}_{y_1}(x) = +\overline{X}_1 x = 1x = x\big|_{x=0} = 0\big|_{x=2} = 2\ \text{m (linear function)},$$

$$\overline{M}^{I}_{y_2}(x) = 0.$$

Portion II – II $(0 < x < 2\text{ m})$

$$M_{yF}^{II}(x) = -Fx + qx^2/2 = -10x + 5x^2 = 5x^2 - 10x\big|_{x=0} = 0\big|_{x=2} = 2\ \text{kNm}$$

(parabola),

$$\overline{M}_{y_1}^{II}(x) = 0,$$

$$\overline{M}_{y_2}^{II}(x) = 0.$$

Portion III – III $(0 < x < 2\text{ m})$

$$M_{yF}^{III}(x) = -M - Fa + qa^2/2 = -10 - 20 + 20 = -10\ \text{kNm},$$

$$\overline{M}_{y_1}^{III}(x) = +\overline{X}_1(a + x) = 1(2 + x) = 2 + x\big|_{x=0} = 2\big|_{x=2} = 4\ \text{m (linear function)},$$

$$\overline{M}_{y_2}^{III}(x) = +\overline{X}_2 x = x\big|_{x=0} = 0\big|_{x=2} = 2\ \text{m (linear function)}.$$

Portion IV – IV $(0 < x < 2\text{ m})$

$$M_{yF}^{IV}(x) = +M - M - F(a - x) + qa(a/2 - x) = -10(2 - x) + 20(1 - x) =$$

$$= -20 + 10x + 20 - 20x = -10x\big|_{x=0} = 0\big|_{x=2} = -20\ \text{kNm (linear function)},$$

$$\overline{M}_{y_1}^{IV}(x) = +\overline{X}_1 2a = 4\ \text{m},$$

$$\overline{M}_{y_2}^{IV}(x) = +\overline{X}_2 a = 2\ \text{m}.$$

To simplify further solution, rewrite the equations inside the Table.

Table

Number of the portion:	Length, m	$M_{yF}(x)$, kNm	$\overline{M}_{y_1}(x)$, m	$\overline{M}_{y_2}(x)$, m
I-I	$0 < x < 2$	-10	x	0
II-II	$0 < x < 2$	$5(x^2 - 2x) =$ $= 5x^2 - 10x$	0	0
III-III	$0 < x < 2$	-10	$2 + x$	x
IV-IV	$0 < x < 2$	$-10x$	4	2

(6) Designing the graphs of bending moments for the force and 2 unit systems (see Fig. 4).

(7) Calculating the coefficients of canonical equations using Mohr's method.

$$\delta_{11}=\frac{1}{EI}\left(\int_0^2 x^2dx+\int_0^2(2+x)^2dx+\int_0^2 16dx\right)=\frac{1}{EI}\left(\int_0^2 x^2dx+\int_0^2(x^2+4x+4)dx+16\int_0^2 dx\right)=$$

$$=\frac{1}{EI}\left(\frac{x^3}{3}+\frac{x^3}{3}+\frac{4x^2}{2}+4x+16x\right)\Bigg|_0^2=\frac{1}{EI}\left(\frac{8}{3}+\frac{8}{3}+8+8+32\right)=\frac{1}{EI}\left(\frac{16}{3}+48\right)=\frac{160}{3EI}.$$

$$\delta_{12}=\delta_{21}=\frac{1}{EI}\left(\int_0^2(2+x)xdx+\int_0^2 8dx\right)=\frac{1}{EI}\left(\int_0^2(2x+x^2)dx+8\int_0^2 dx\right)=$$

$$=\frac{1}{EI}\left(\frac{2x^2}{2}+\frac{x^3}{3}+8x\right)\Bigg|_0^2=\frac{1}{EI}\left(4+\frac{8}{3}+16\right)=\frac{1}{EI}\left(20+\frac{8}{3}\right)=\frac{68}{3EI}.$$

$$\delta_{22}=\frac{1}{EI}\left(\int_0^2 x^2dx+\int_0^2 4dx\right)=\frac{1}{EI}\left(\frac{x^3}{3}+4x\right)\Bigg|_0^2=\frac{1}{EI}\left(\frac{8}{3}+8\right)=\frac{32}{3EI}.$$

$$\Delta_{1F}=\frac{1}{EI}\left(\int_0^2(-10x)dx+\int_0^2-10(2+x)dx+\int_0^2(-40x)dx\right)=$$

$$=\frac{1}{EI}\left(-10\int_0^2 xdx-10\int_0^2(2+x)dx-40\int_0^2 xdx\right)=\frac{1}{EI}\left(-\frac{10x^2}{2}-10\left(2x+\frac{x^2}{2}\right)-\frac{40x^2}{2}\right)\Bigg|_0^2=$$

$$=\frac{1}{EI}\left(-5x^2-20x-5x^2-20x^2\right)\Big|_0^2=\frac{1}{EI}(-20-40-20-80)=-\frac{160}{EI}.$$

$$\Delta_{2F}=\frac{1}{EI}\left(\int_0^2(-10x)dx+\int_0^2-20xdx\right)=\frac{1}{EI}\left(-10\int_0^2 xdx-20\int_0^2 xdx\right)=$$

$$=\frac{1}{EI}\left(-\frac{10x^2}{2}-\frac{20x^2}{2}\right)\Bigg|_0^2=\frac{1}{EI}\left(-5x^2-10x^2\right)\Big|_0^2=\frac{1}{EI}(-20-40)=-\frac{60}{EI}.$$

(8) Calculating the coefficients of canonical equations using graphical method.

$$\Delta_{1F} = \frac{1}{EI}\left((-10\times 2)\times(+1) + (0) + (-10\times 2)\times(+3) + \left(-\frac{20\times 2}{2}\right)\times(+4) \right) = -\frac{160}{EI},$$

$$\Delta_{2F} = \frac{1}{EI}\left((0) + (0) + (-10\times 2)\times(+1) + \left(-\frac{20\times 2}{2}\right)\times(+2) \right) = -\frac{60}{EI},$$

$$\delta_{11} = \frac{1}{EI}\left(\left(+\frac{4\times 4}{2}\right)\times\left(+\frac{2}{3}\times 4\right) + (+4\times 2)\times(+4) \right) = \frac{160}{3EI},$$

$$\delta_{22} = \frac{1}{EI}\left(\left(+\frac{2\times 2}{2}\right)\times\left(+\frac{2}{3}\times 2\right) + (+2\times 2)\times(+2) \right) = \frac{32}{3EI},$$

$$\delta_{12} = \frac{1}{EI}\left(\left(+\frac{2\times 2}{2}\right)\times(0) + (+2\times 2)\times(+1) + \left(+\frac{2\times 2}{2}\right)\times\left(+\frac{2}{3}\times 2\right) + (+4\times 2)\times(+2) \right) = \frac{68}{3EI}.$$

$$\delta_{21} = \delta_{12} = \frac{68}{3EI}.$$

(9) Substituting the coefficients into canonical equations to find X_1 and X_2.

a) First canonical equation is:

$$\delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1F} = 0,$$

$$\frac{160}{3EI}X_1 + \frac{68}{3EI}X_2 - \frac{160}{EI} = 0,$$

$$\frac{160}{3}X_1 + \frac{68}{3}X_2 - 160 = 0,$$

$$160X_1 + 68X_2 - 480 = 0,$$

$$40X_1 + 17X_2 - 120 = 0.$$

$$X_1 = \frac{120 - 17X_2}{40}. \qquad (*)$$

b) Second canonical equation is:

$$\delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2F} = 0,$$

$$\frac{68}{3EI}X_1+\frac{32}{3EI}X_2-\frac{60}{EI}=0,$$

$$\frac{68}{3}X_1+\frac{32}{3}X_2-60=0,$$

$$68X_1+32X_2-180=0,$$

$$17X_1+8X_2-45=0.$$

Substituting the value of X_1 from (*) into last equation leads to

$$17\left(\frac{120-17X_2}{40}\right)+8X_2=45,$$

$$\frac{17}{40}(120-17X_2)+8X_2=45,$$

$$\frac{2040}{40}-\frac{289X_2}{40}+8X_2=45,$$

$$2040-289X_2+320X_2=1800,$$

$$31X_2=-240,$$

$$X_2=-\frac{240}{31}=-7.742 \text{ kNm.}$$

After substituting the X_2 value in equation (*) we have

$$X_1=\frac{120-17\left(-\frac{240}{31}\right)}{40}=\frac{120+\frac{4080}{31}}{40}=\frac{\frac{3720+4080}{31}}{40}=\frac{7800}{31\cdot 40}=\frac{7800}{1240}=6.290 \text{ kNm.}$$

As the value of X_2 is negative, its original direction in equivalent system must be changed on opposite.

Conclusion: static indeterminacy is opened.

(10) Calculating the internal forces in statically determinate equivalent system shown on Fig 5.

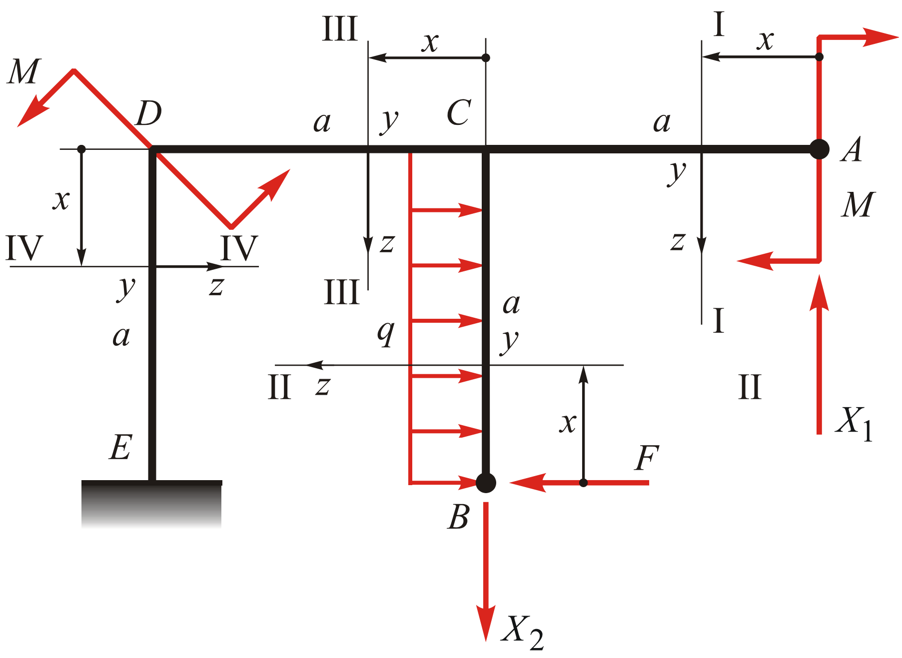


Fig. 5

Portion I – I $(0 < x < 2\text{ m})$

$N_x^I(x) = 0$ kN,

$Q_z^I(x) = -X_1 = -6.29$ kN,

$M_y^I(x) = +X_1 x - M = 6.29x - 10\big|_{x=0} = -10\big|_{x=2} = 2.58$ kNm (linear function).

Portion II – II $(0 < x < 2\text{ m})$

$N_x^{II}(x) = +X_2 = 7.742$ kN,

$Q_z^{II}(x) = +F - qx = 10 - 10x\big|_{x=0} = 10\big|_{x=2} = -10$ kNm (linear function),

$M_y^{II}(x) = -Fx + qx^2/2 = -10x + 5x^2\Big|_{x=0} = 0\big|_{x=2} = -20 + 20 = 0$ kNm

(parabola).

Note, that shear force graph intersects the x axis. In such case maximum bending moment should be found:

(a) The cross-section of maximal moment is determined by equating to zero the shear force equation:

$Q_z^{II}(x_e)=0$, $F-qx_e=0$, $qx_e=F$, $x_e=1$ m.

(b) The value of maximal moment is determined by substituting the $x_e=1$ m into the bending moment equation:

$$M_y^{II}(x_e)=-10x_e+5x_e^2=-10+5=-5 \text{ kNm.}$$

Portion III – III $(0<x<2\text{ m})$

$N_x^{III}(x)=-F+qa=-10+20=10$ kN,

$Q_z^{III}(x)=-X_1+X_2=-6.29+7.742=1.452$ kN,

$M_y^{III}(x)=-M+X_1(a+x)-X_2x-Fa+qa^2/2=$

$=-10+6.29(2+x)-7.742x-20+20=-10+12.58+6.29x-7.742x=$

$=-1{,}452x+2{,}58\big|_{x=0}=2{,}58\big|_{x=2}=-0{,}314$ kNm (linear function).

Portion IV – IV $(0<x<2\text{ m})$

$N_x^{IV}(x)=X_1-X_2=6.29-7.742=-1.452$ kN,

$Q_z^{IV}(x)=-F+qa=-10+20=10$ kN,

$M_y^{IV}(x)=+M-M+2aX_1-aX_2-F(a-x)+qa(a/2-x)=$

$$=6.29\times4-7.742\times2-10(2-x)+20(1-x)=25.16-15.484-20+10x+20-20x=$$

$$=-10x+9.676\big|_{x=0}=9.676\big|_{x=2}=-10.324 \text{ kNm.}$$

(11) Designing of graphs of bending moment and also shear and normal force distributions in the equivalent system.

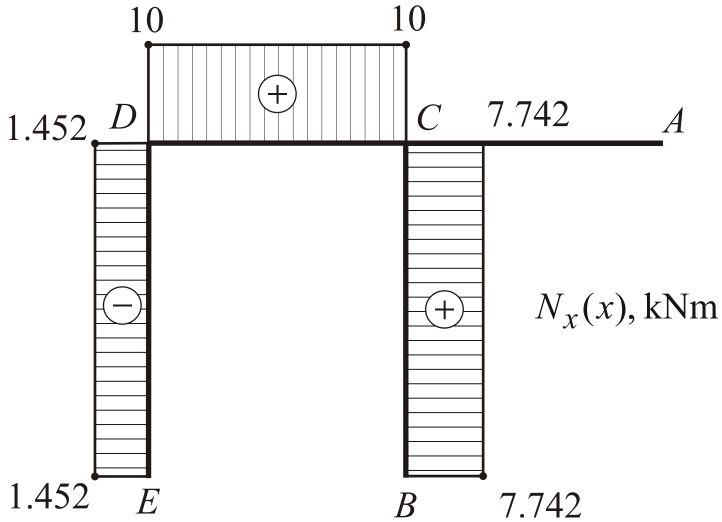


Fig. 6a

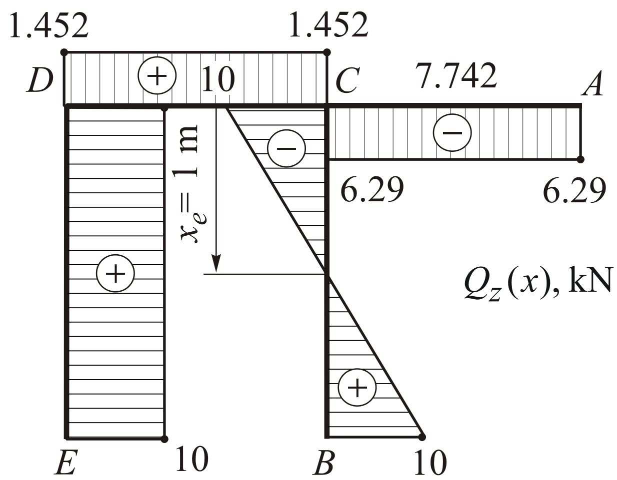


Fig. 6b

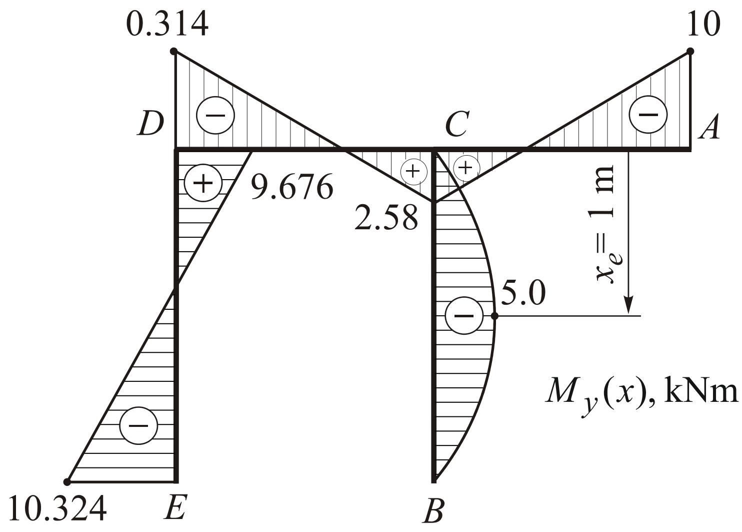


Fig. 6c

(12) Checking the equilibrium in two rod connections

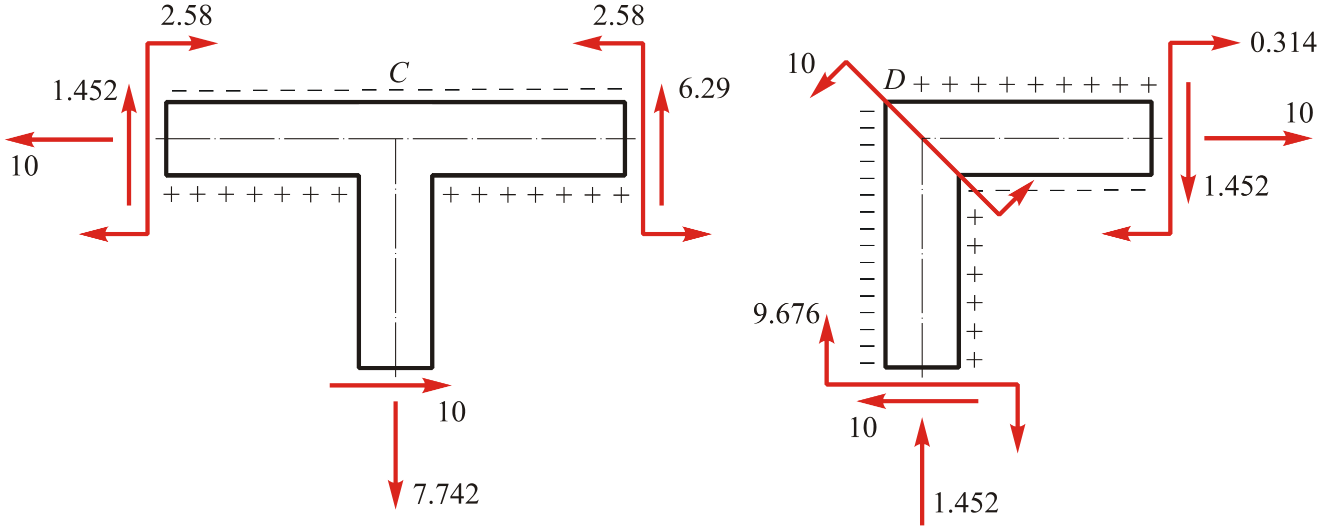


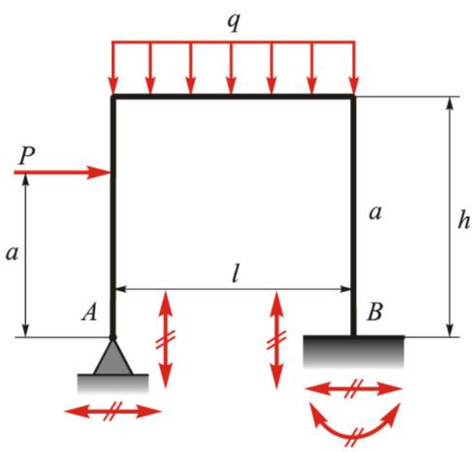
Fig. 7

Example 4 Two-fold statically indeterminate frame (example of home problem)

National aerospace university
"Kharkiv Aviation Institute"
Department of aircraft strength

Subject: mechanics of materials
Document: home problem
Topic: Internal Forces in Statically Indeterminate Plane Frames.
Full name of the student, group

Variant: 145 **Complexity: 1**



Given: $q = 20$ kN/m, $P = 20$ kN, $h = 3$ m, $l = 4$ m, $a = 2$ m.
Goal: 1) open static indeterminacy using the force method and draw the graphs $N_x(x), Q_z(x), M_y(x)$.

signature
Full name of the lecturer

Mark:

Solution

1. Determining the degree of static indeterminacy.

According to formula, degree of static indeterminacy is $K = m - n$, where m is total number of unknown reactions; n is the number of equations of static.

After substituting,

$K = 5 - 3 = 2$.

Conclusion: plane frame is 2-fold statically indeterminate.

2. Drawing the frame in scale:

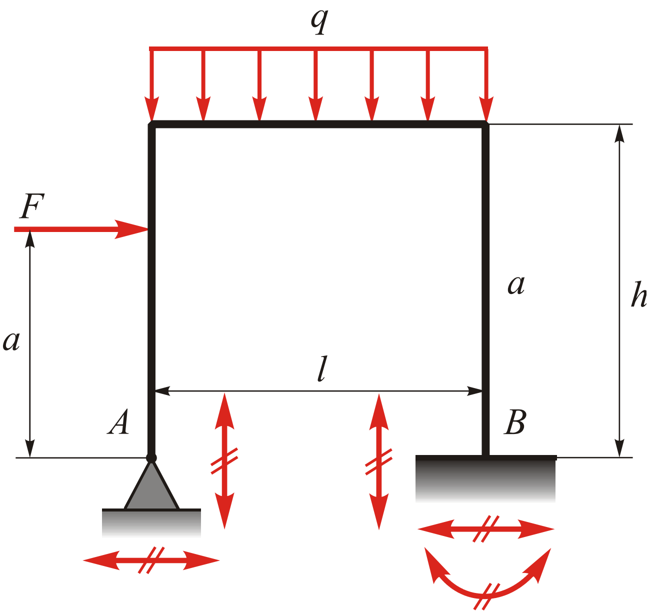


Fig. 1 Plane frame in equilibrium under external loading and reactions

3. Selecting the base system by omitting external forces and reactions of redundant constraints. The base system should be statically determinate. It is shown on Fig. 2.

4. Creating the equivalent system. The effect of omitted constraints is replaced by their reactions: the reactive moment in B support X_1 and the horizontal reaction X_2. Besides, remained reactions R_{A_v}, R_{A_h}, R_{B_v} are applied to the equivalent system in an arbitrary directions. Equivalent system is shown on Fig. 3.

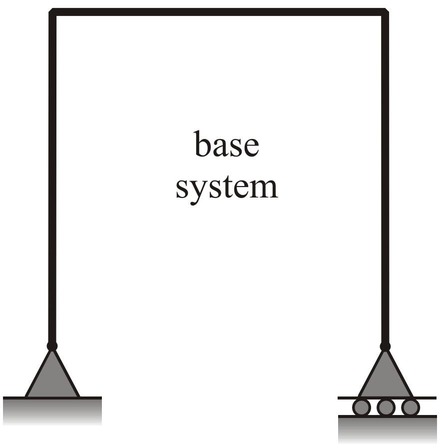


Fig. 2 Selected base system

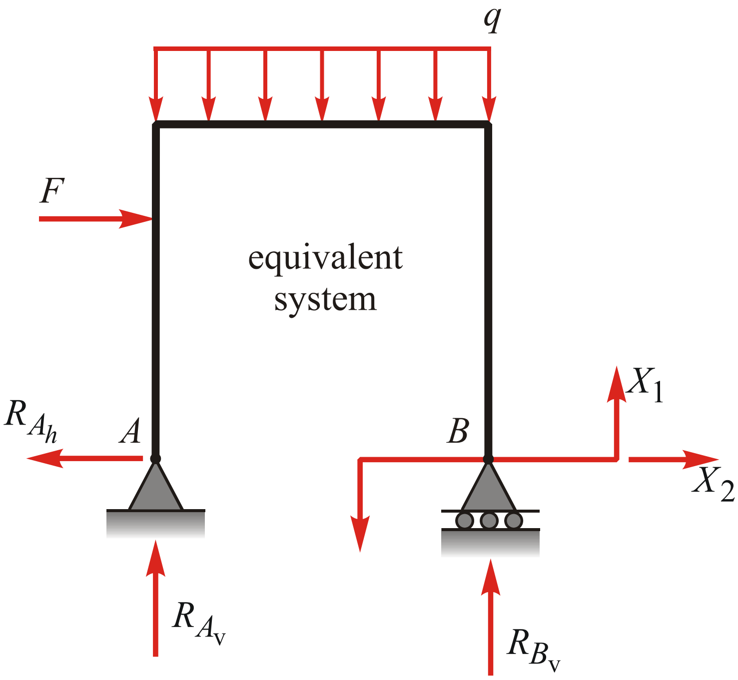


Fig. 3 Selected equivalent system

5. Designing the system of canonical equations (compatibility equations). From the view-point of evident zero angle of *B*-section rotation and horizontal displacement of *B* point in selected equivalent system is the following:

$$\begin{cases} \theta_B(X_1, X_2, F) = 0, \\ \delta_{hor_B}(X_1, X_2, F) = 0. \end{cases} \quad \text{or, in canonical shape} \quad \begin{cases} \delta_{11}X_1 + \delta_{12}X_2 + \Delta_{1F} = 0, \\ \delta_{21}X_1 + \delta_{22}X_2 + \Delta_{2F} = 0. \end{cases}$$

6. Calculating the coefficients of canonical equations.

To find 6 coefficients δ_{11}, δ_{12}, Δ_{1F}, δ_{21}, δ_{22}, Δ_{2F}, let us create a force (*F*) system and 2 unit systems: (1) and (2). These systems are shown on Fig. 4.

(a) Let us preliminary calculate the unknown reactions in all three systems using the equations of static:

– for the force system:

$$\sum F_x = 0; \quad F - R_{A_h} = 0 \to R_{A_h} = F = +20\,\text{kN},$$

$$\sum M_A = 0; \quad R_{B_v} l - \frac{ql^2}{2} - Fa = 0 \to R_{B_v} = +50\,\text{kN},$$

$$\sum F_y = 0; \quad -ql - R_{A_v} + R_{B_v} = 0 \to R_{A_v} = -30\,\text{kN}.$$

– for the first unit system:

$$\sum F_x = 0; \quad R_{A_h},$$

$$\sum M_A = 0; \quad \bar{R}_{B_v} l - \bar{M} = 0 \to \bar{R}_{B_v} = +\frac{1}{4},$$

$$\sum F_y = 0; \quad +\bar{R}_{A_v} + \bar{R}_{B_v} = 0 \to \bar{R}_{A_v} = -\frac{1}{4}.$$

– for the second unit system:

$$\sum F_x = 0; \quad -\bar{\bar{R}}_{A_h} + 1 = 0 \to \bar{\bar{R}}_{A_h} = +1,$$

$$\sum M_A = 0; \quad \bar{\bar{R}}_{B_v} l = 0 \to \bar{\bar{R}}_{B_v} = 0,$$

$$\sum F_y = 0; \quad -\bar{\bar{R}}_{A_v} + \bar{\bar{R}}_{B_v} = 0 \to \bar{\bar{R}}_{A_v} = 0.$$

Note, that negative reactions should be changed in their direction before writing the equations of bending moments. Actual reactions are shown on Fig. 4.

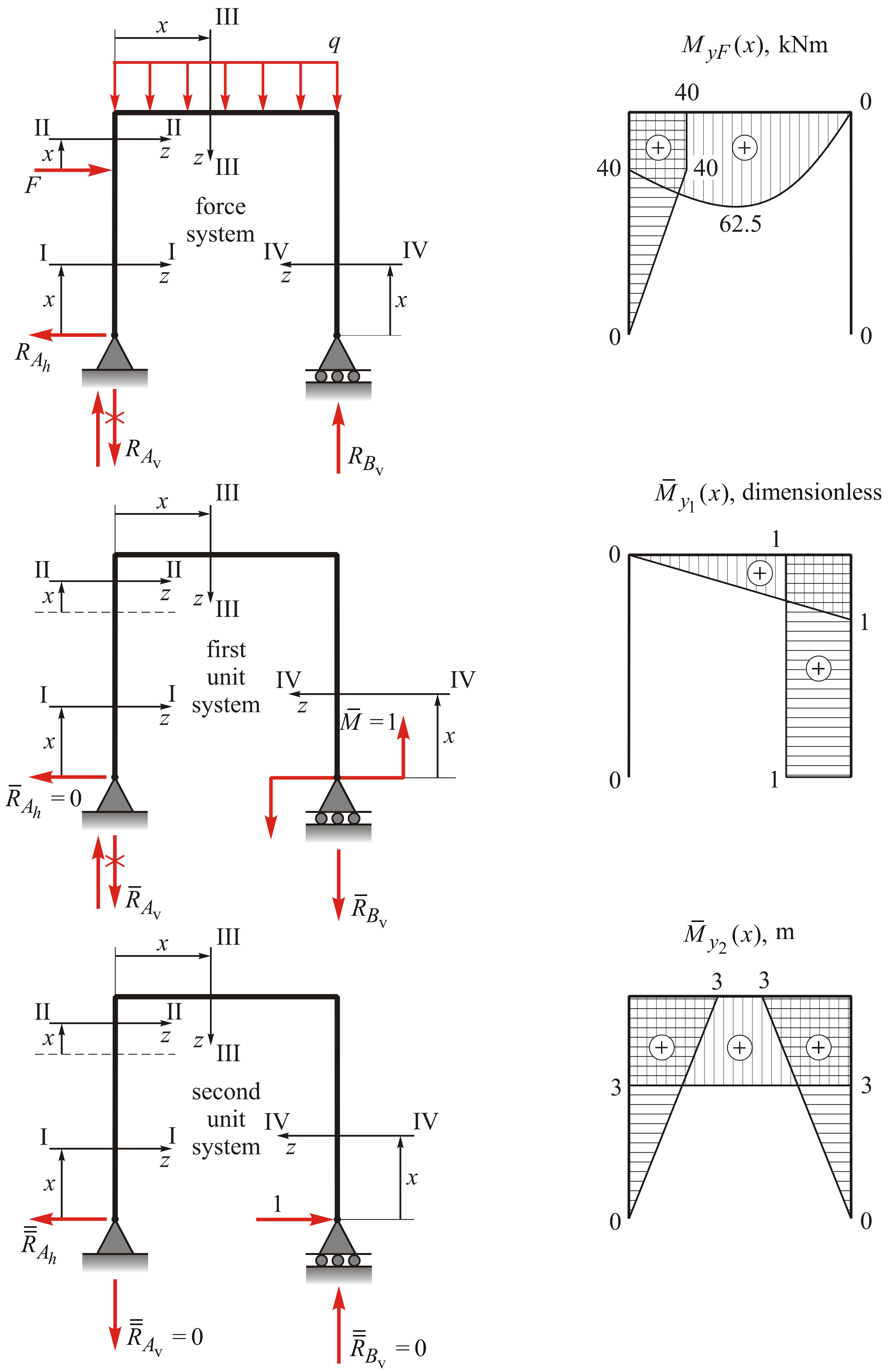


Fig. 4 Force system and two unit systems with corresponding graphs of bending moments

(b) Now we define internal forces and construct internal force factors diagrams (see Fig. 4). The point is that the primary displacements in the frame are induced by bending. Hence, neglecting the shear and tension-compression deformations in the bars, we draw diagrams for the bending moment due to the given forces F and q in the force system and also for two unit systems. For this purpose, we compile the Table 1 of bending moments according to the given portions.

Table 1

Number of the portion:	Length, m	$M_{yF}(x)$, kNm	$\overline{M}_{y1}(x)$, m	$\overline{M}_{y2}(x)$, m
I-I	$0<x<2$	$R_{A_h}x=20x$	0	$\bar{\bar{R}}_{A_h}x=x$
II-II	$0<x<1$	$R_{A_h}(a+x)-Fx=40$	0	$\bar{\bar{R}}_{A_h}(a+x)=2+x$
III-III	$0<x<4$	$-R_{A_v}x+R_{A_h}h-F(h-a)-\frac{qx^2}{2}=40+30x+10x^2$	$\bar{R}_{A_v}x=\frac{x}{4}$	$\bar{\bar{R}}_{A_h}h=3$
IV-IV	$0<x<3$	0	1	x

(c) The coefficients of canonical equations are determined by the Mohr's method and are checked by graphical solution:

$$\Delta_{1F}=\frac{1}{EI_y}\left[\frac{1}{4}\int_0^4\left(40+30x-10x^2\right)xdx\right]=\frac{1}{4EI_y}\left[\left(\frac{40x^2}{2}+\frac{30x^3}{3}-\frac{10x^4}{4}\right)\Bigg|_0^4\right]=$$

$$=\frac{1}{4EI_y}[320+640-640]=+\frac{80}{EI_y},$$

$$\Delta_{2F}=\frac{1}{EI_y}\left[\int_0^2 20x^2dx+\int_0^1 40(2+x)dx+\int_0^4 3\left(40+30x-10x^2\right)dx\right]=$$

$$=\frac{1}{EI_y}\left[\frac{20x^3}{3}\Bigg|_0^2+\left(80x+20x^2\right)\Big|_0^1+\left(120x+\frac{90x^2}{2}-10x^3\right)\Bigg|_0^4\right]=$$

$$=\frac{1}{EI_y}\left[\frac{160}{3}+100+480+640\right]=+\frac{713.3}{EI_y},$$

$$\delta_{11} = \frac{1}{EI_y}\left[\int_0^2 0dx + \int_0^1 0dx + \int_0^4 \left(\frac{x}{4}\right)\left(\frac{x}{4}\right)dx + \int_0^3 1dx\right] = +\frac{13}{3EI_y},$$

$$\delta_{22} = \frac{1}{EI_y}\left[\int_0^2 xxdx + \int_0^1 (2+x)(2+x)dx + \int_0^4 3\cdot 3dx + \int_0^3 xxdx\right] = +\frac{54}{EI_y},$$

$$\delta_{12} = \delta_{21} = \frac{1}{EI_y}\left[\int_0^2 0dx + \int_0^1 0dx + \int_0^4 3\frac{x}{4}dx + \int_0^3 xdx\right] = +\frac{21}{2EI_y}.$$

Checking by graphical solution:

$$\delta_{11} = \frac{1}{EI_y}\left[\left(\frac{1\times 4}{2}\right)\left(\frac{2}{3}\right) + (3\times 1)(1)\right] = \frac{13}{3EI_y},$$

$$\delta_{12} = \delta_{21} = \frac{1}{EI_y}\left[\left(\frac{3\times 3}{2}\right)(0) + (3\times 4)\left(\frac{1}{2}\right) + \left(\frac{3\times 3}{2}\right)(1)\right] = +\frac{21}{2EI_y},$$

$$\delta_{22} = \frac{1}{EI_y}\left[\left(\frac{3\times 3}{2}\right)\left(\frac{2}{3}\times 3\right) + (3\times 4)(3) + \left(\frac{3\times 3}{2}\right)\left(\frac{2}{3}\times 3\right)\right] = +\frac{54}{EI_y}.$$

7. Calculating the reactions of redundant constrains X_1 and X_2 in equivalent system.

After substituting the coefficients into the canonical equations we obtain:

$$\begin{cases} \frac{13}{3EI_y}X_1 + \frac{10.5}{EI_y}X_2 + \frac{80}{EI_y} = 0, \\ \frac{10.5}{EI_y}X_1 + \frac{54}{EI_y}X_2 + \frac{713.3}{EI_y} = 0. \end{cases} \rightarrow \begin{cases} X_1 = +25.6\ \text{kNm}, \\ X_2 = -18.2\ \text{kN}. \end{cases}$$

This completes the opening of static indeterminacy since unknown reactions X_1 and X_2 in equivalent system are found. Three remaining reactions will be calculated from equations of static.

8. Calculating the internal forces in equivalent system and drawing their diagrams.

(a) Preliminary calculating the reactions of supports using the equations of static.

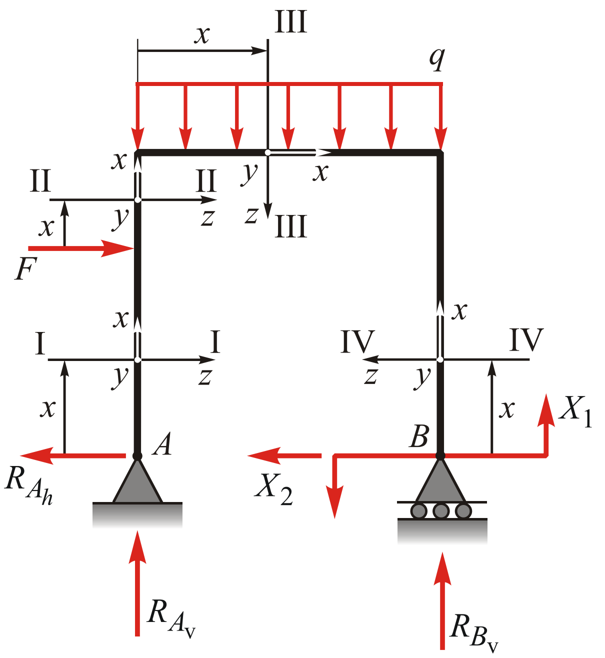


Fig. 5 Equivalent system after finding X_1 and X_2. They are shown in actual directions.

$\sum F_x = 0$:

$F - X_2 - R_{A_h} = 0 \to$

$\to R_{A_h} = F - X_2 = 20 - 18.2 = +1.8\,\text{kN}$,

$\sum M_A = 0$:

$$R_{B_v} l + X_1 - \frac{ql^2}{2} - Fa = 0 \to$$

$$\to R_{B_v} = \frac{1}{4}\left[20 \times 2 + \frac{20 \times 16}{2} - 25.6\right] =,$$

$= +43.6\,\text{kN}$

$\sum F_y = 0$:

$R_{A_v} - ql + R_{B_v} = 0 \to$

$\to R_{A_v} = 20 \times 4 - 43.6 = +36.4\,\text{kN}$.

Note. Positive signs of calculated reactions correspond to their actual directions.

(b) Writing the equations of the functions $N_x(x)$, $Q_z(x)$ and $M_y(x)$ in an arbitrary sections of each portion:

Portion I-I: $0 \le x \le a$

$N_x(x) = -R_{A_v} = -36.4\,\text{kN}$,

$Q_z(x) = R_{A_h} = +1.8\,\text{kN}$,

$$M_y(x) = R_{A_h} x = 1.8x\Big|_{x=0} = 0\Big|_{x=2} = +3.6\,\text{kNm}.$$

Portion II-II: $0 \le x \le (h-a)$

$N_x(x) = -R_{A_v} = -36.4\,\text{kN}$,

$Q_z(x) = R_{A_h} - F = 1.8 - 20 = -18.2\,\text{kN}$,

$$M_y(x) = R_{A_h}(a + x) - Fx = 3.6 + 1.8x - 20x\Big|_{x=0} = 3.6\Big|_{x=1} = -14.6\,\text{kNm}.$$

Portion III-III: $0 \le x \le l$

$$N_x(x) = R_{A_h} - F = 1.8 - 20 = -18.2 \text{ kN},$$

$$Q_z(x) = R_{A_v} - qx \Big|_{x=0} = 36.4 \Big|_{x=4} = -43.6 \text{ kN}.$$

Note, that the change of $Q_z(x)$ function sign from "+" to "–" predict bending moment extreme value within the boundaries of the potion. Let us find the coordinate of the cross-section with extreme bending moment by equating to zero shear force function:

$$Q_z(x_e) = R_{A_v} - qx_e = 0 \rightarrow x_e = \frac{R_{A_v}}{q} = \frac{36.4}{20} = 1.82 \text{ m}.$$

$$M_y(x) = R_{A_v} x + R_{A_h} h - F\left(h - a\right) - \frac{qx^2}{2} =$$

$$= 36.4x - 14.6 - 10x^2 \Big|_{x=0} = -14.6 \Big|_{x=4} = -29 \Big|_{x_e=1.82} = +18.5 \text{ kNm}.$$

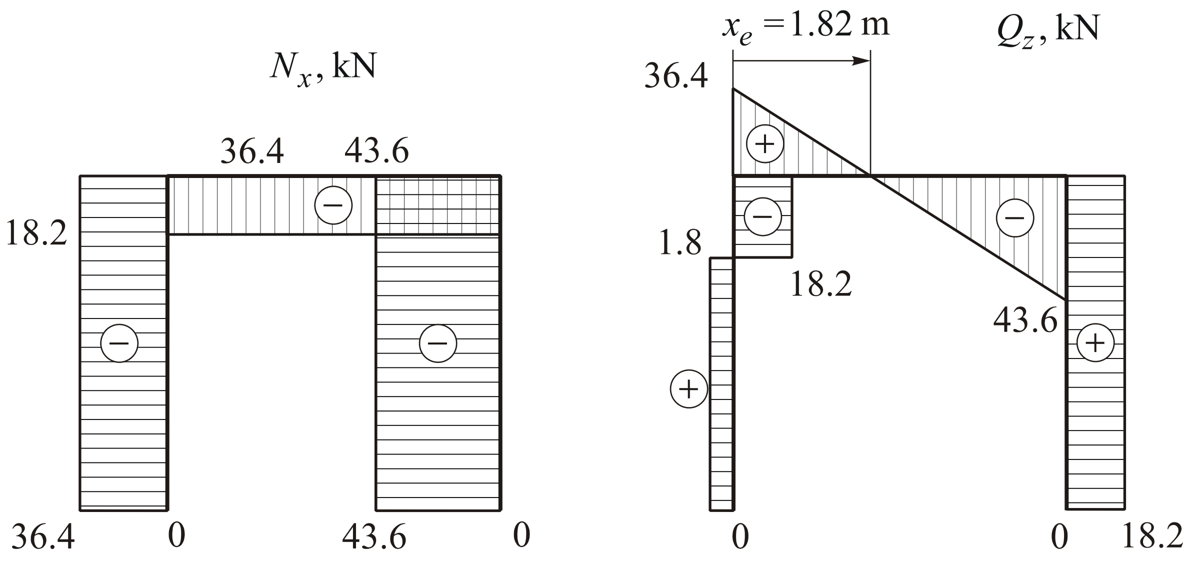
Portion IV-IV: $0 \le x \le h$,

$$N_x(x) = -R_B = -43.6 \text{ kN},$$

$$Q_z(x) = X_2 = +18.2 \text{ kN},$$

$$M_y(x) = -X_2 x + X_1 = -18.2x + 25.6 \Big|_{x=0} = 25.6 \Big|_{x=h} = -29 \text{ kNm}.$$

Corresponding graphs are shown in Fig. 6.



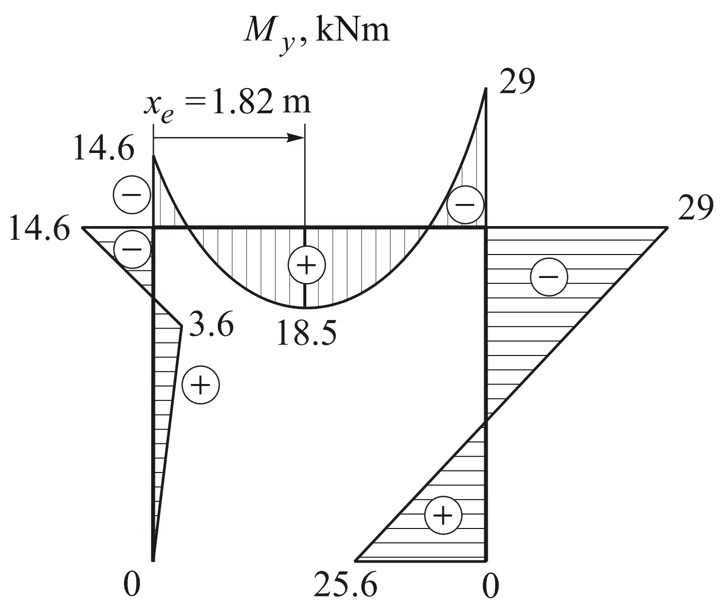


Fig. 6 Internal forces in given statically indeterminate frame

9. Checking the results.

(a) Checking the equilibrium of two infinitely small segments of the frame.

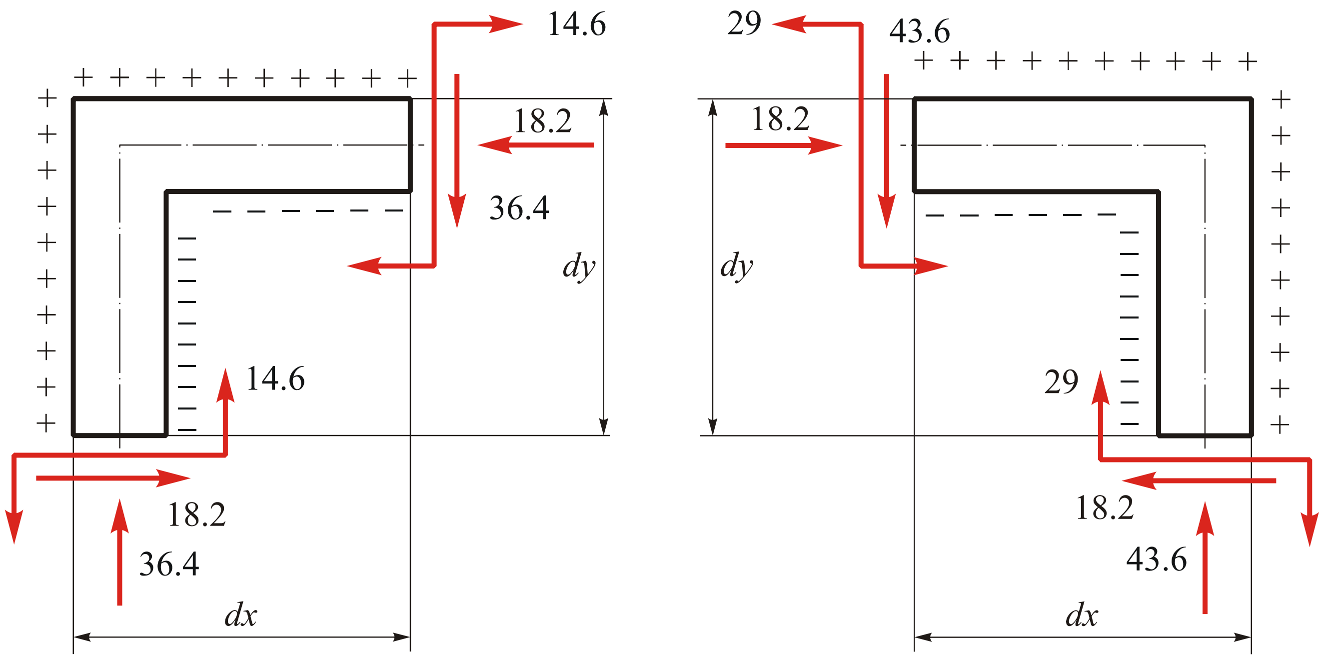


Fig. 7

(b) Calculating the evidently zero vertical displacement of *A* point.

For this purpose, we should select, first of all, new base system (see Fig. 8).

In our case, we selected the base system in which really immobile in vertical and horizontal directions left support A is free. As has been noted earlier, it is statically determined. Corresponding equivalent system is shown on Fig. 9. Due to static equivalence of both equivalent systems (compare, please, Figs. 3, 9), reactions X_3 and X_4 in left support A are known from our solution: $X_3 = 1.8\,\text{kN}$, $X_4 = 36.4\,\text{kN}$. Next evident feature of new equivalent system is in evidently zero linear displacements of A point. It means that future calculating the vertical displacement must lead to zero result. For this purpose, according to the Mohr's method, new equivalent system will be considered as the force system, and unit system will be designed applying vertical unit dimensionless force in A point. Both these systems are shown in Figs. 10 and 11.

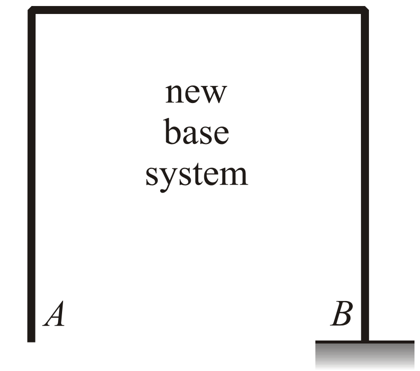


Fig. 8

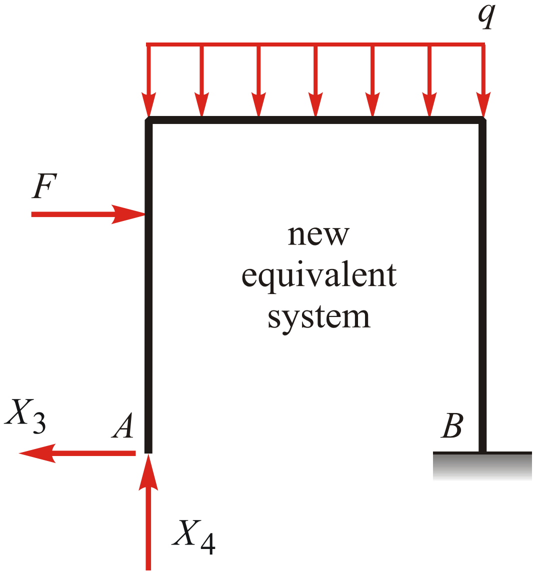


Fig. 9

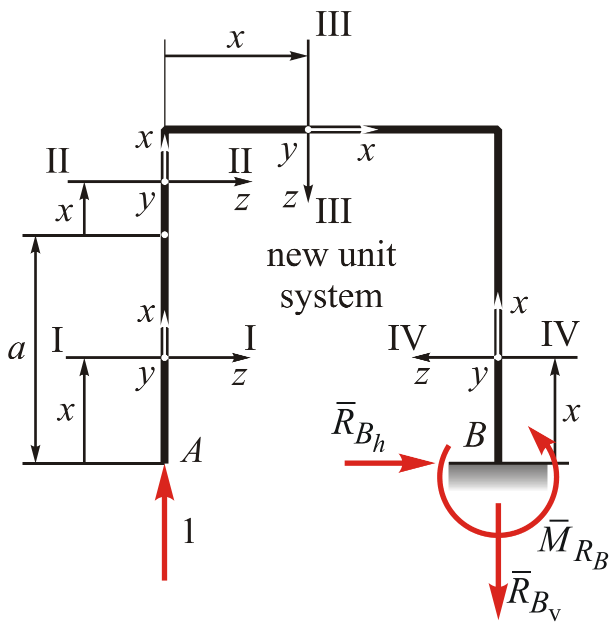


Fig. 10

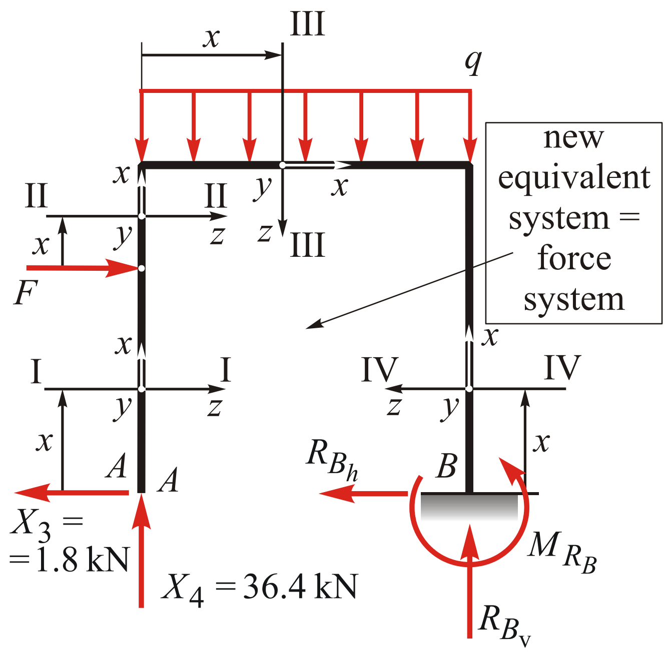


Fig. 11

Before multiplying bending moment equations of the unit and force systems in the Mohr's integral, it is necessary to calculate the reactions in rigid support of both systems. For the force system, these reactions are known from previous solution: $R_{B_h} = 18.2\,\text{kN}$, $R_{B_v} = 43.6\,\text{kN}$, $M_{R_B} = 25.6\,\text{kNm}$. For the unit system, the reactions are calculated from the equations of equilibrium:

$$\sum F_x = 0 \rightarrow \bar{R}_{B_h},$$
$$\sum F_y = 0 = 1 - \bar{R}_{B_v} \rightarrow \bar{R}_{B_v} = 1,$$
$$\sum M_B = 0 = 1 \cdot l - \bar{M}_{R_B} \rightarrow \bar{M}_{R_B} = 4\text{ m}.$$

Bending moment equations of the force and unit systems are introduced into Table 2.

Table 2

Number of the portion:	Length, m	$M_{yF}(x)$, kNm	$\overline{M}_y(x)$, m
I-I	$0 < x < 2$	$X_3 x = 1.8x$	0
II-II	$0 < x < 1$	$X_3(a + x) - Fx = 3.6 - 18.2x$	0
III-III	$0 < x < 4$	$-F(h-a) - \frac{qx^2}{2} + X_4 x + X_3 h =$ $= -10x^2 + 36.4x - 14.6$	x
IV-IV	$0 < x < 3$	$M_{R_B} - R_{B_h} x = 25.6 - 18.2x$	4

Vertical linear displacement of A point is:

$$\delta_{A_v} = \frac{1}{EI_y}\left[\int_0^2 0dx + \int_0^1 0dx + \int_0^4 x\left(36.4x - 14.6 - 10x^2\right)dx + \int_0^3 4\left(25.6 - 18.2x\right)dx\right] =$$
$$= \frac{1}{EI_y}\left[\frac{36.4x^3}{3}\Bigg|_0^4 - \frac{14.6x^2}{2}\Bigg|_0^4 - \frac{10x^4}{4}\Bigg|_0^4 + 102.4x\Bigg|_0^3 - 72.8\frac{x^2}{2}\Bigg|_0^3\right] =$$
$$= \frac{1}{EI_y}\left[776.53 - 116.8 - 640 + 307.2 - 327.6\right] = \frac{1}{EI_y}(A - B) = \frac{1}{EI_y}(1083.73 - 1084.40).$$

Error of calculating:

$$\Delta = \frac{(A-B)}{A} \cdot 100\% = \frac{1083.73 - 1084.40}{1083.73} \cdot 100\% = 0.0618\% \approx 0.$$