LECTURE 24 Continuous (Multispan) Beams and the Method of Three Moments

Beams that have more than one span and there are continuous throughout their lengths (Fig. 1) are known as **continuous beams**. They are commonly encountered in aircraft, bridges, buildings, pipelines and various kinds of specialized structures. Continuous beams are statically indeterminate and may be analyzed by the method of superposition.

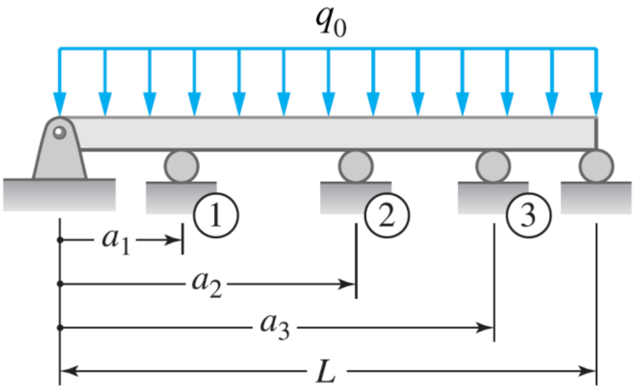


Fig. 1

In this lecture, we describe a particular form of the superposition method, called the **method of three moments**, that is especially useful in the analysis of the continuous beams.

We assume that all loads acting on the beam are vertical and that there are no restraints against rotation at the supports (that is, none of the supports is fixed or clamped). In addition, we assume that there are no axial deformations due to restraints against horizontal movement. Under these conditions, all reactions at the supports are vertical forces.

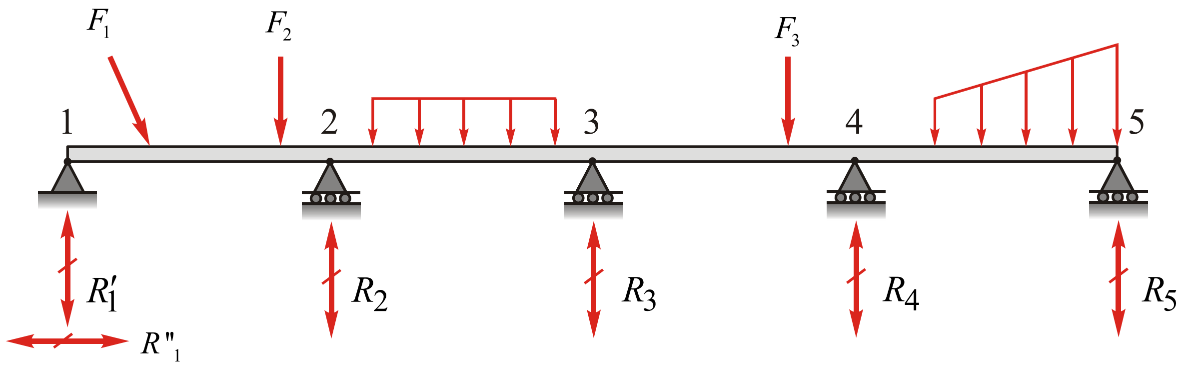


Fig. 2

The design scheme of continuous beam is represented in Fig. 3

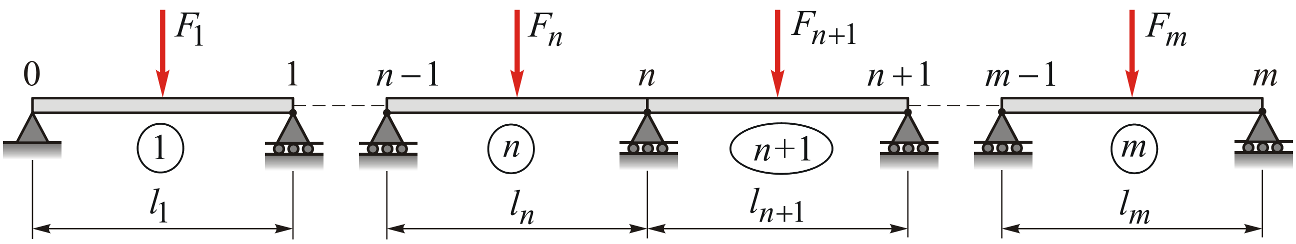


Fig. 3

A system may be singly, two-fold, three-fold,...., *m*-fold statically indeterminate, depending on the number of **redundant supports**.

Consider the *n* und *n*+1 **spans**. The moments M_{n-1}, M_n and M_{n+1} are applied to replace the removed constraints between adjacent spans:

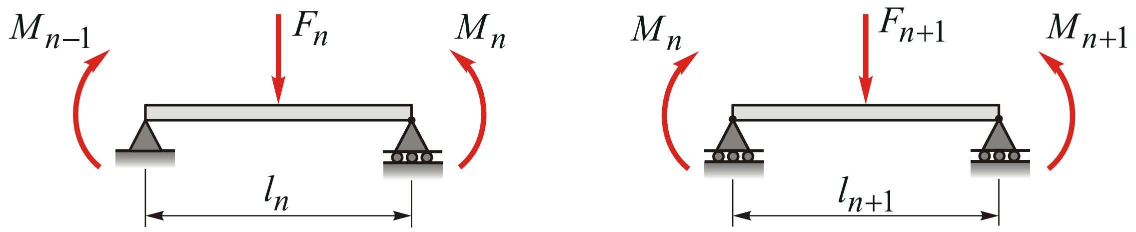


Fig. 4

The moments shown in Fig. 4 are considered positive. The **displacement equation** expresses the fact that the ***mutual angle of rotation of the adjacent sections of the spans n and n+1 over the n-th support must be zero*** (Fig. 5):

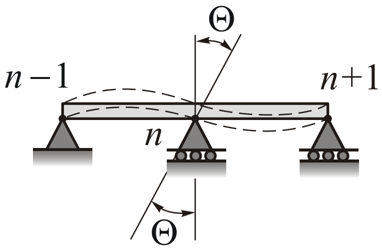


Fig. 5

$$\left|\Theta_{n,n}\right| = \left|\Theta_{n,n+1}\right|. \tag{1}$$

(a) ***Consider the n span*** and determine the $\Theta_{n,n}$ angle:

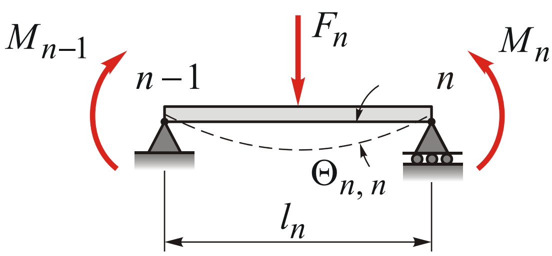


Fig. 6

Evidently, that $\Theta_{n,n} = \Theta_{n,n}(F_n) + \Theta_{n,n}(M_{n-1}) + \Theta_{n,n}(M_n).$ (2)

Determine these angles. For this purpose we draw moment diagrams due to the given forces. The rigidity *EI* is assumed to be the same for all the spans:

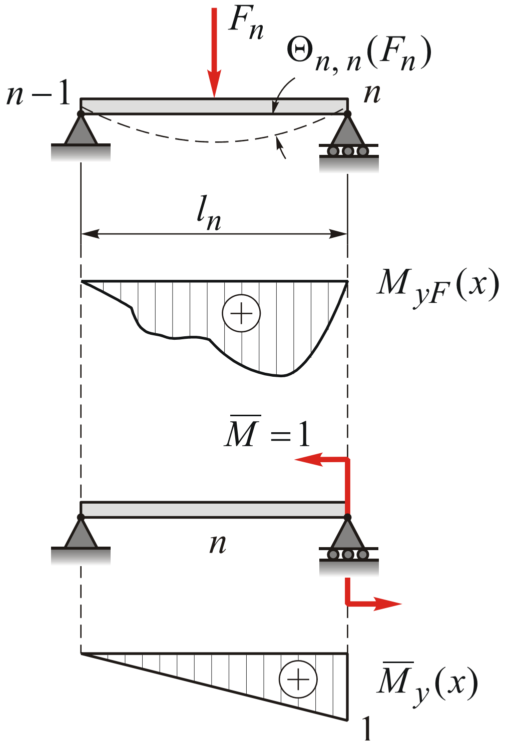


Fig. 7

In accordance with the Mohr's method:

$$\Theta_{n,n}(F_n)=\frac{1}{EI_y}\int_0^{l_n} M_{yF}(x)\overline{M}_y(x)dx=\alpha_{n,n}\,. \quad (3)$$

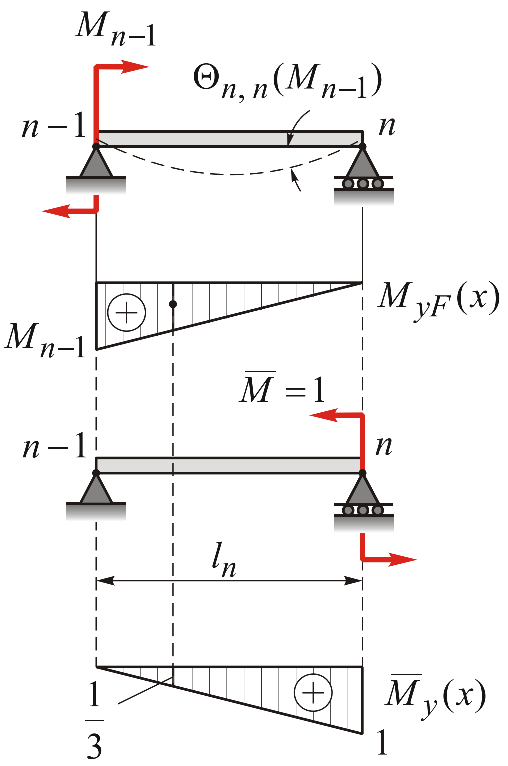


Fig. 8

To calculate the $\Theta_{n,n}(M_{n-1})$ angle we will use the Vereschagin's method:

$$\Theta_{n,n}(M_{n-1})=\frac{1}{EI}\left(\frac{M_{n-1}l_n}{2}\right)\left(\frac{1}{3}\right). \quad (4)$$

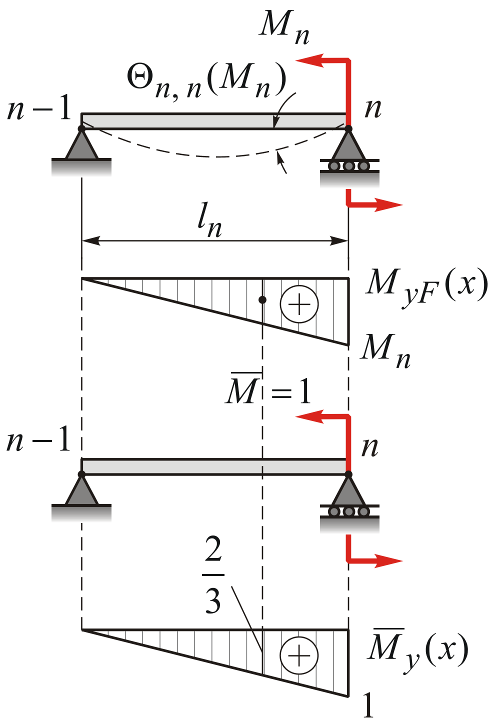


Fig. 9

To calculate the $\Theta_{n,n}(M_n)$ angle we will use the Veresschagin's method:

$$\Theta_{n,n}(M_n)=\frac{1}{EI}\left(\frac{M_n l_n}{2}\right)\left(\frac{2}{3}\right). \qquad (5)$$

Substitute the results obtained in expression (2):

$$\Theta_{n,n}=\alpha_{n,n}+\frac{M_{n-1}l_n}{6EI_y}+\frac{M_n l_n}{3EI_y}. \qquad (6)$$

(b) ***Consider the n+1 span*** and determine the $\Theta_{n,n+1}$ angle:

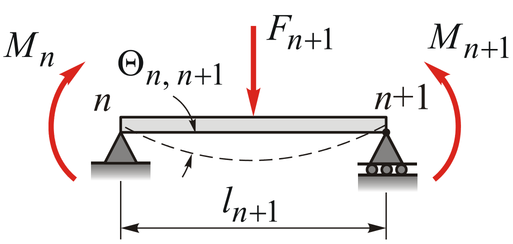


Fig. 10

Evidently, that

$$\Theta_{n,n+1}=\Theta_{n,n+1}(F_{n+1})+\Theta_{n,n+1}(M_n)+$$
$$+\Theta_{n,n+1}(M_{n+1}). \qquad (7)$$

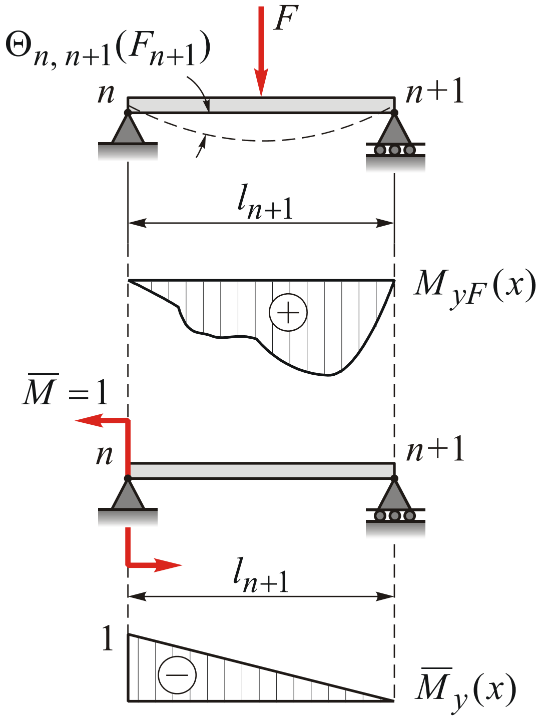


Fig. 11

In accordance with the Mohr's method

$$\Theta_{n,n+1}(F_{n+1}) = \frac{1}{EI}\int_0^{l_n+1} M_{yF}(x)\overline{M}_y(x)dx =$$

$$= -\alpha_{n,n+1}. \tag{8}$$

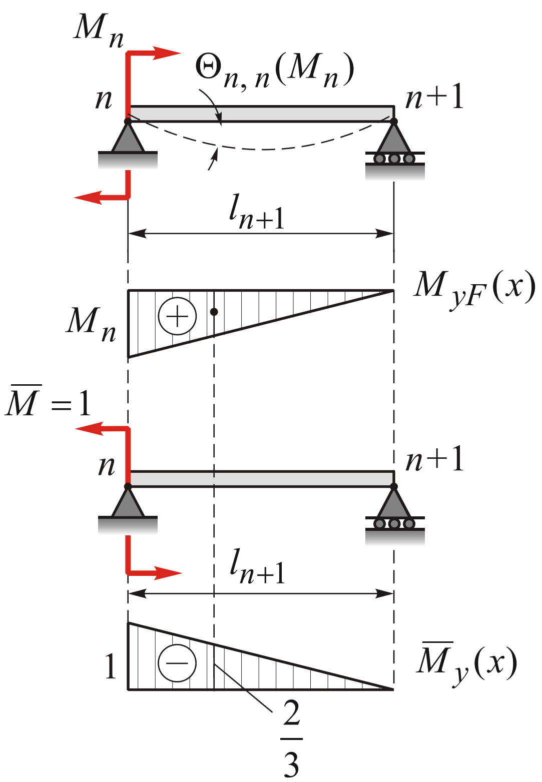


Fig. 12

To calculate the $\Theta_{n,n+1}(M_n)$ angle we will use the Vereschagin's method:

$$\Theta_{n,n+1}(M_n) = \frac{1}{EI_y}\left(\frac{M_n l_{n+1}}{2}\right)\left(-\frac{2}{3}\right). \tag{9}$$

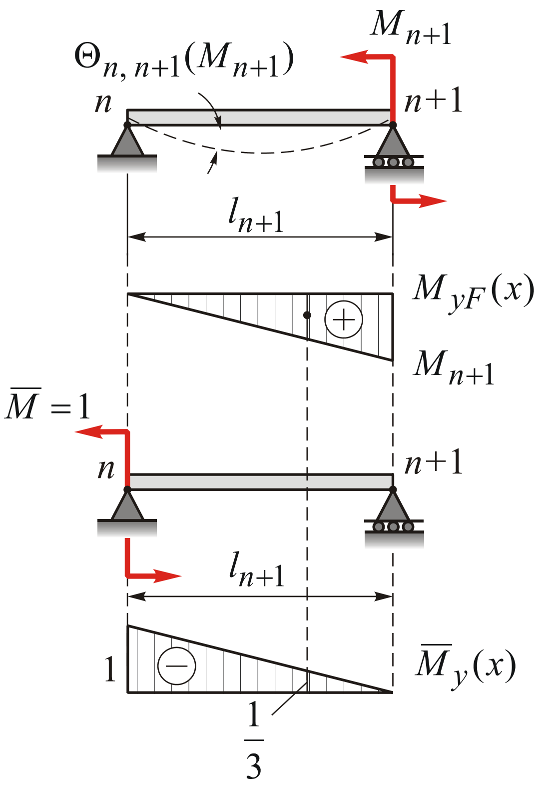


Fig. 13

To calculate the $\Theta_{n,n+1}(M_n)$ angle we will use the Vereschagin's method:

$$\Theta_{n,n+1}(M_{n+1})=\frac{1}{EI_y}\left(\frac{M_{n+1}l_{n+1}}{2}\right)\left(-\frac{1}{3}\right). \quad (10)$$

We substitute the results obtained into expression (7)

$$\Theta_{n,n+1}=\left(-\alpha_{n,n+1}\right)+\left(-\frac{M_n l_{n+1}}{3EI_y}\right)+\left(-\frac{M_{n+1}l_{n+1}}{6EI_y}\right). \quad (11)$$

Equation (1) takes the form

$$\alpha_{n,n}+\frac{M_{n-1}l_n}{6EI_y}+\frac{M_n l_n}{3EI_y}=-\alpha_{n,n+1}-\frac{M_n l_{n+1}}{3EI_y}-\frac{M_{n+1}l_{n+1}}{6EI_y}. \quad (12)$$

The equation assumes the following final form

$$M_{n-1}l_n+2M_n\left(l_n+l_{n+1}\right)+M_{n+1}l_{n+1}=-6EI_y\left(\alpha_{n,n}+\alpha_{n,n+1}\right). \quad (13)$$

This equation is known as the **equation of three moments**. The principle of deriving such equations for a multispan beam is sufficiently clear. ***The equation of three moments is set up for each pair of adjacent spans*** with all pairs considered in succession. ***Consequently the number of equations for a multispan beam is equal to the degree of static indeterminacy***.

After the equations have been solved and the moments found, it is an easy matter to draw a bending moment diagram and to find the stresses in the beam.

Example 1 Open the statically indeterminacy of the beam shown in Fig. 14.

Given: F=10 kN, q=10 kN/m, $M_A = 20\text{kNm}$, $M_B = 40\text{kNm}$, $M_C = 30\text{kNm}$, a=1m, b=2m, c=3m, d=1m

$Q_z(x), M_y(x) - ?$

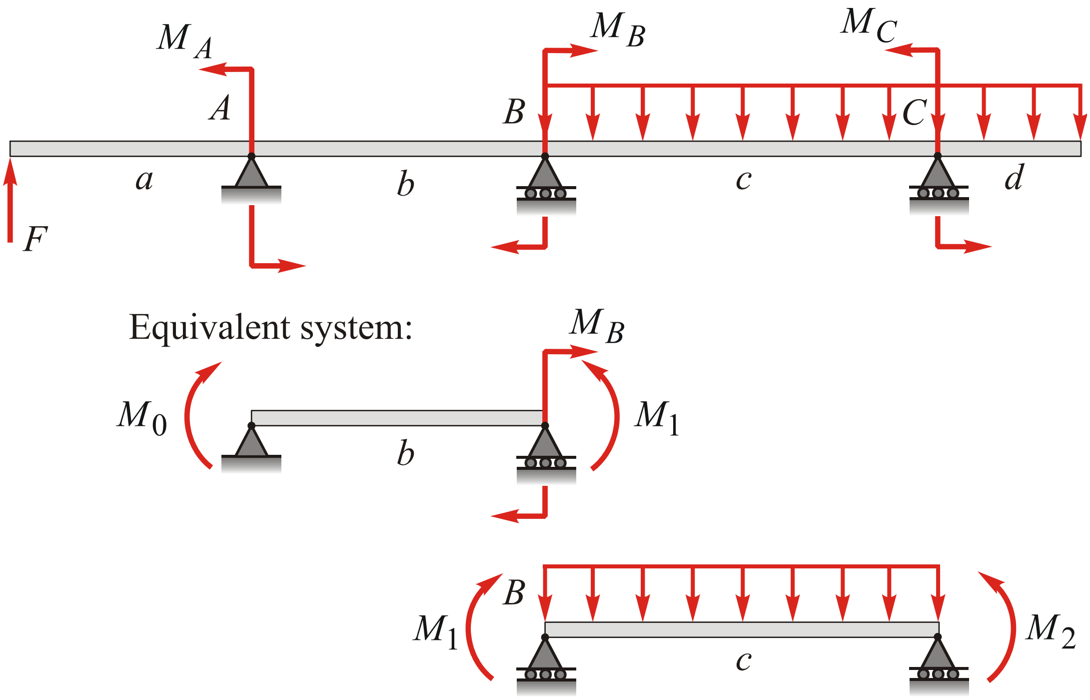


Fig. 14

In this case

$$M_0 = Fa - M_A = 10\times 1 - 20 = -10\text{kNm},$$

$$\mathrm{M}_2 = -q\frac{d^2}{2} + M_C = -10\frac{1}{2} + 30 = +25\text{kNm}.$$

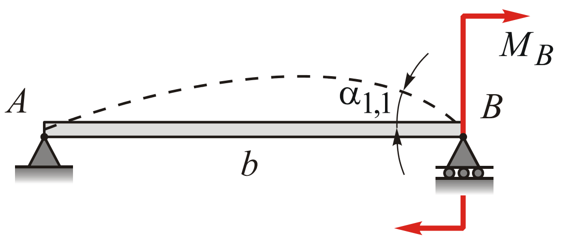


Fig. 15

$$\alpha_{1.1} = \alpha_{11}(M_B) = -\frac{M_B b}{3EI} = -\frac{40\times 2}{3EI} = -\frac{80}{3EI}.$$

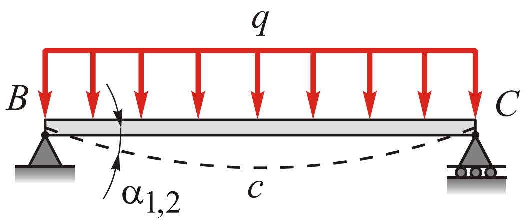


Fig. 16

$$\alpha_{1,2} = \alpha_{1,2}(q) = +\frac{qc^3}{24EI} = +\frac{10\times 3^3}{24EI} = +\frac{45}{4EI}$$

Substituting into equation (13) we get

$$-10\times 2+2M_1(2+3)+25\times 3=-6EI\left(-\frac{80}{3EI}+\frac{45}{4EI}\right).$$

From this solution, $M_1=+3.75$ kNm.

Therefore, opening of static indeterminacy is finished and we will consider the equilibrium of two separate spans:

(a) ***left span*****:**

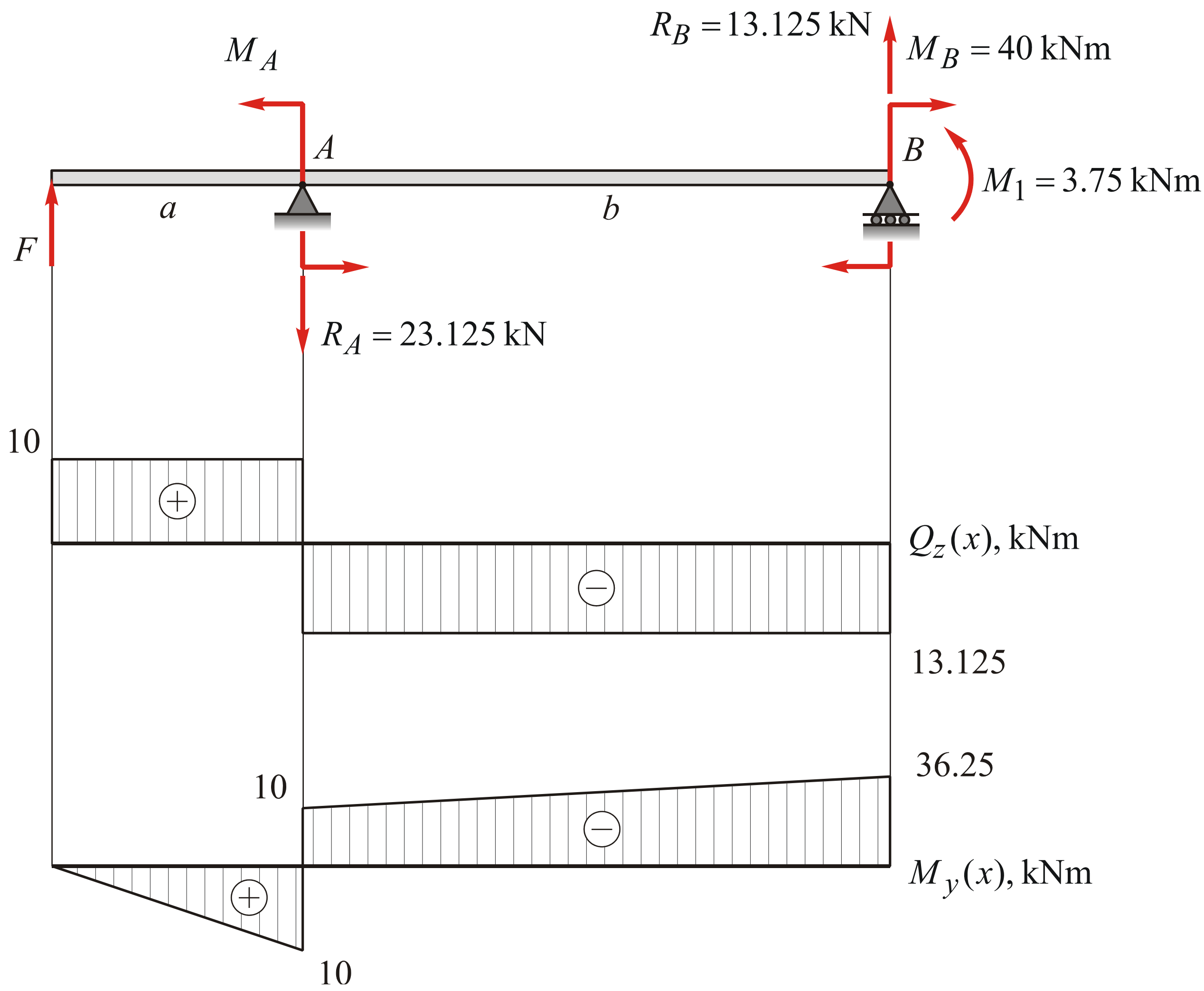


Fig. 17

(b) ***right span*:**

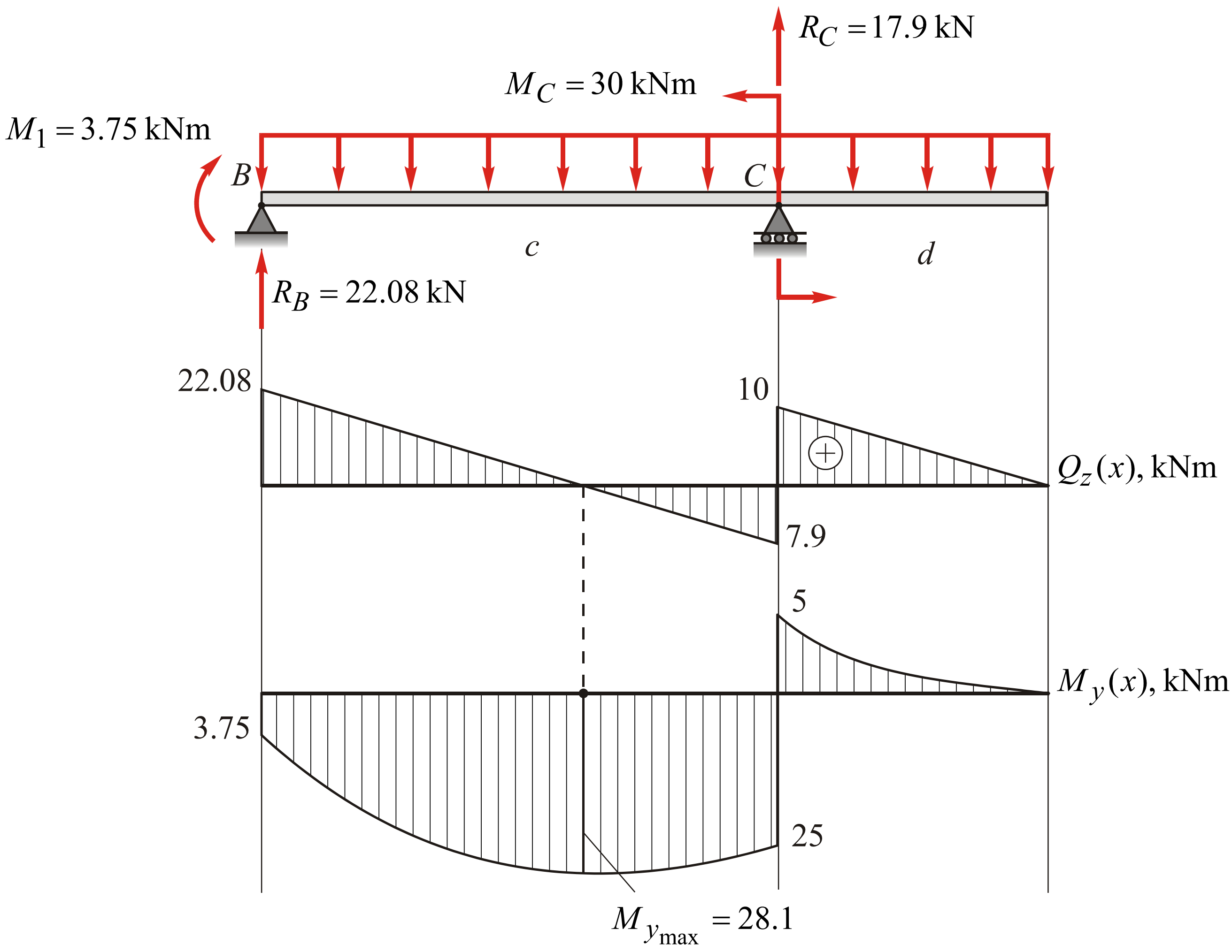


Fig. 18

Example 2 Open the static indeterminacy of the beam shown in Fig. 19.

Given: F=10 kN, a=1m, b=2m, c=3m

$Q_z(x), M_y(x) - ?$

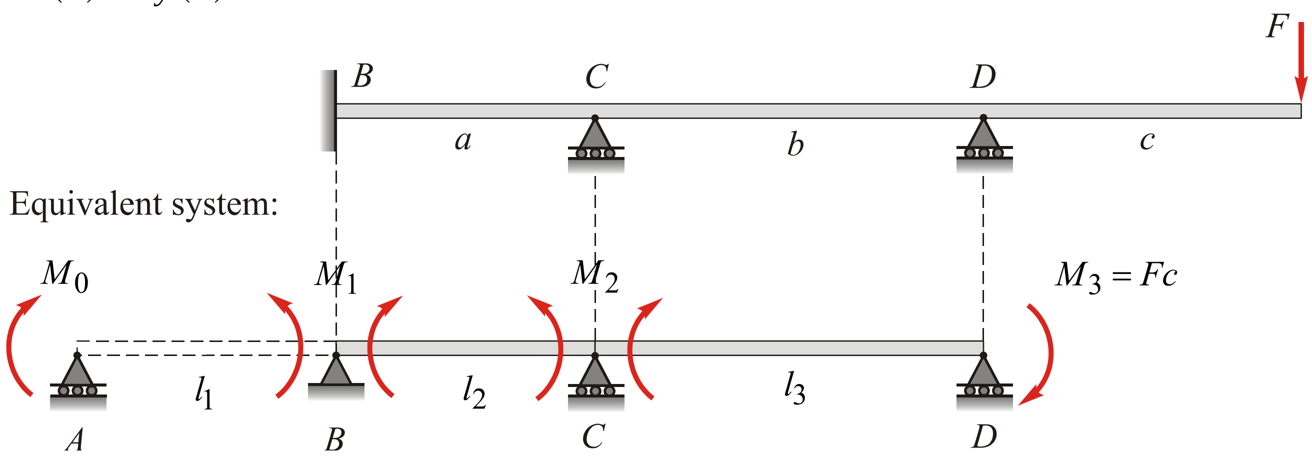


Fig. 19

The system is two-fold statically indeterminate. ***A feature of the system is the presence of the overhanging end on the right and the built-in end on the left.*** We transfer the force F to the point over the right support and introduce the moment M_3 in place of the removed overhang (cantilever part).

We replace the built-in fixation by two infinitely close supports, i.e., we introduce a span of length $l_1 = 0$ on the left. Equivalent system is shown in Fig. 19

For the pair of spans AB and BC equation (13) becomes

$$M_0 l_1 + 2M_1(l_1 + l_2) + M_2 l_2 = -6EI(\alpha_{1,1} + \alpha_{1,2}), \quad M_0 = 0, \ \alpha_{1,1} = \alpha_{1,2} = 0.$$

$$l_1 = 0, \ l_2 = a, \quad l_3 = b.$$

We proceed to the second pair of spans. The moment of the given force $M_D = M_3$ may be considered either as a support moment equal to $-Fc$ or as a given external load. We shall consider the moment $-Fc$ as a support moment. Equation (13) then yields

$$M_1 l_2 + 2M_2(l_2 + l_3) + M_3 l_3 = -6EI(\alpha_{2,2} + \alpha_{2,3}), \quad \alpha_{2,2} = \alpha_{2,3} = 0.$$

By solving the equations obtained simultaneously, we find

$$\begin{cases} 0 \times 0 + 2M_1(0 + a) + M_2 a = 0, \\ M_1 a + 2M_2(a + b) - Fcb = 0, \end{cases}$$

$$M_2 = \frac{2Fbc}{3a + 4b} = 10.91 \text{ kNm}, \quad M_1 = \frac{Fbc}{3a + 4b} = 5.45 \text{ kNm}.$$

Thereafter we draw a bending moment diagram connecting to separately considered spans:

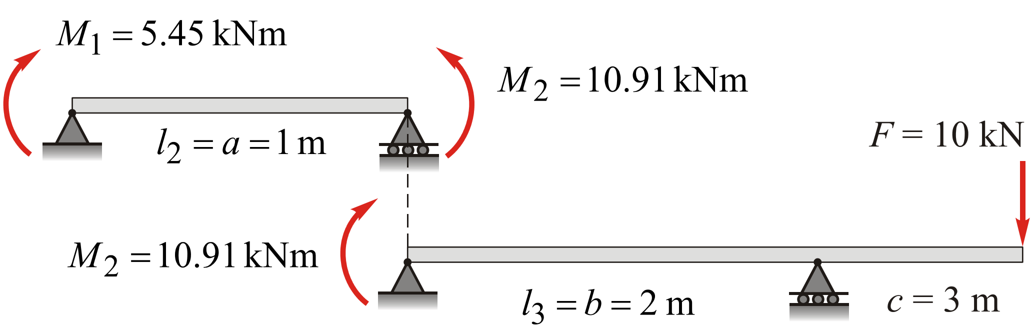


Fig. 20

The graphs of internal forces are shown on Fig. 20. They are designed after the reactions R_B, R'_C (left span), and R''_C, R_D (right span). Note, that actual directions of the reactions are shown on Fig. 20.

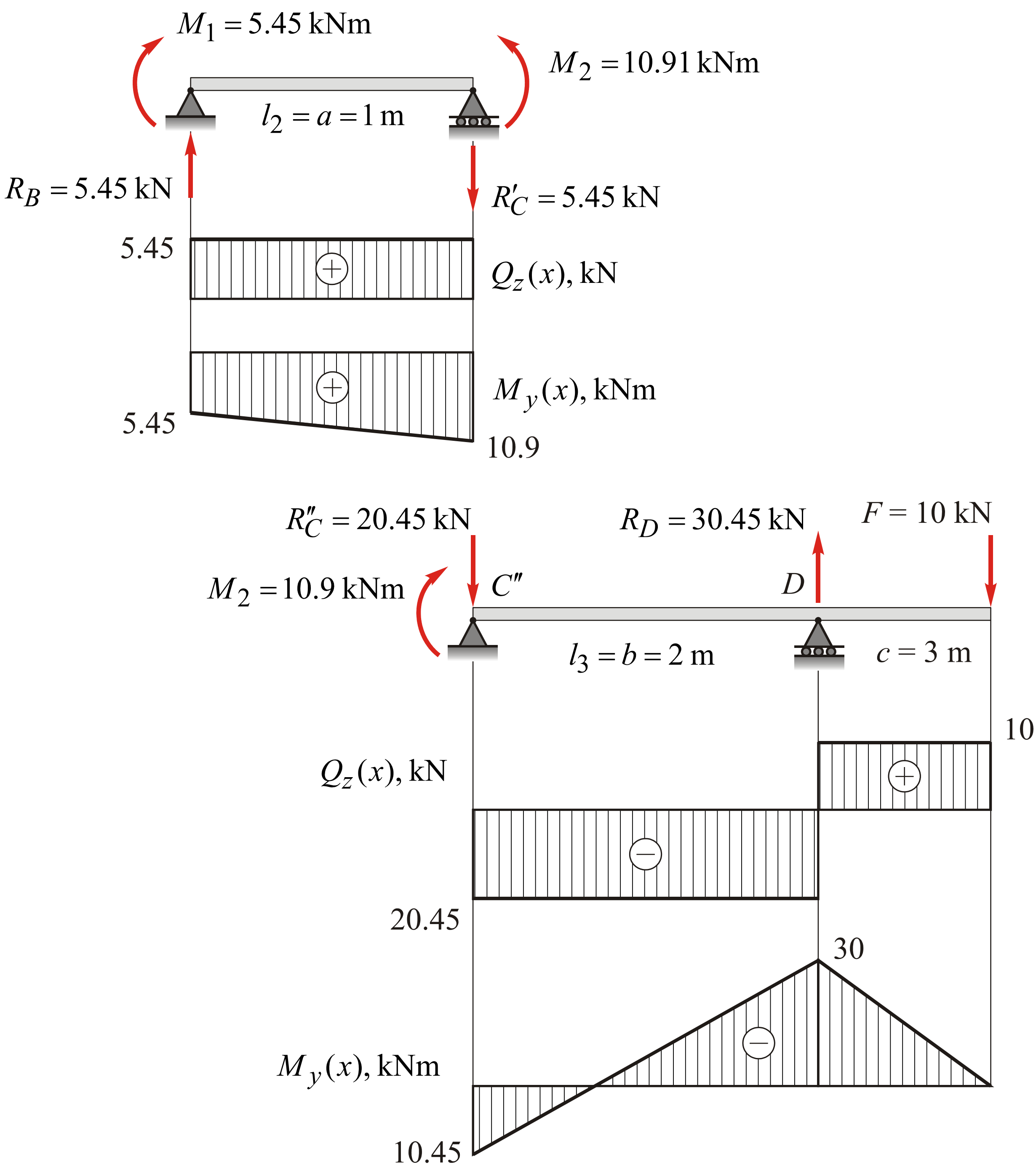


Fig. 21

Example 3 (Home problem)

Given: two-span beam (see Fig. 1), $M = 10\text{kNm}$, $q = 20\text{kN/m}$, $a = 3\text{m}$, $b = 2\text{m}$, $c = 2\text{m}$, $d = 1\text{m}$.

It is necessary:

1) open static indeterminacy using three moment equations and design $M_y(x)$ and $Q_z(x)$ diagrams;

2) open static indeterminacy using force method and design $M_y(x)$ and $Q_z(x)$ diagrams;

3) compare the results.

Solution:

(A) Application of the equation of three moments.

(1) First of all we determine the degree of static indeterminacy according to the formula

$$K = m - n,$$

where K is the degree of static indeterminacy, m is the number of unknown reactions, n is the number of equations of static equilibrium. So, $m = 4; n = 3$ and $K = 1$. The fact of the beam being singly statically indeterminate gets obvious.

(2) Designing the equivalent system (see Fig. 1). It is developed by introducing virtual hinge into mid support cross section and adding into it unknown internal bending moment $M_1 = X_1$. Also, internal bending moments in left and right supports are represented in equivalent system by two concentrated moments M_0 and M_2. They are calculated by applying the method of sections using sign conventions shown on Fig. 2.

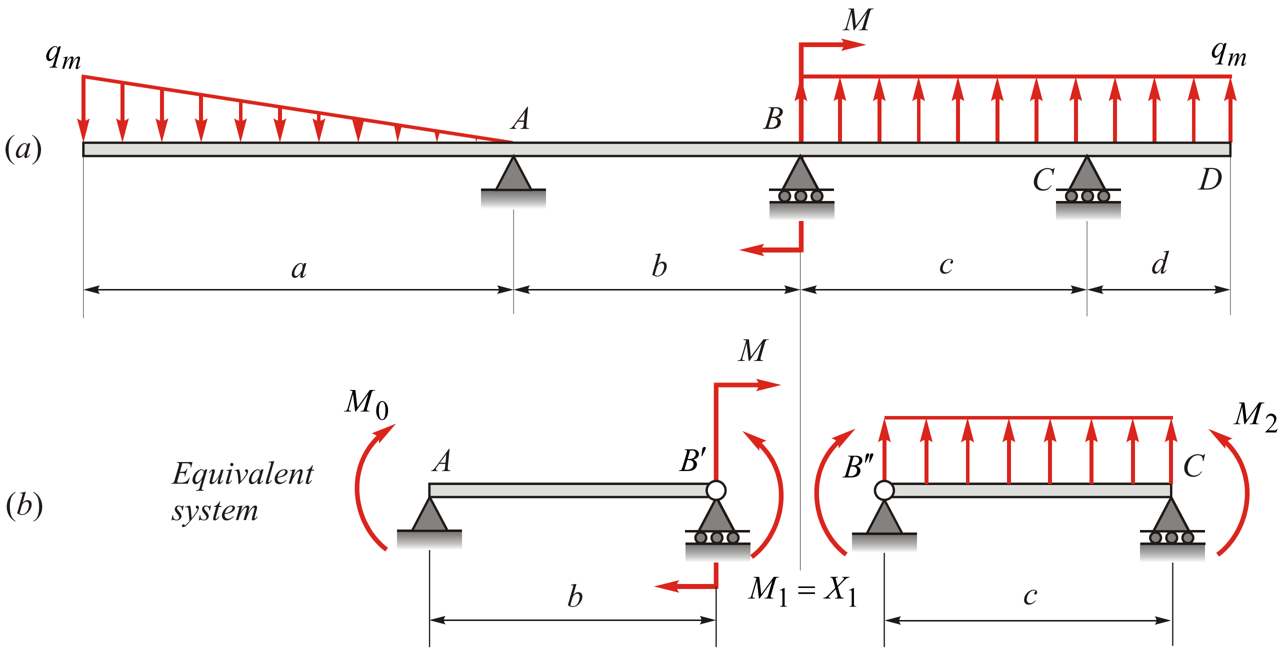


Fig. 1

Note, that the moments M_0, M_1, M_2 are applied in their positive directions according to the sign conventions, shown on Figs. 2, 3.

Sign conventions:

a) for shear forces b) for bending moments

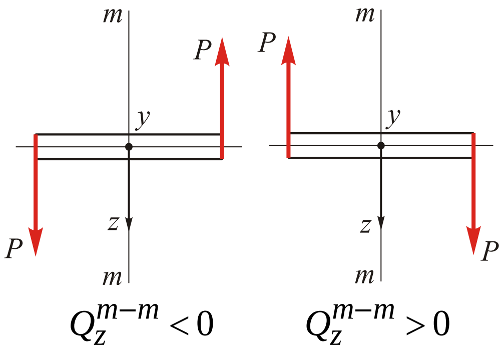


Fig. 2

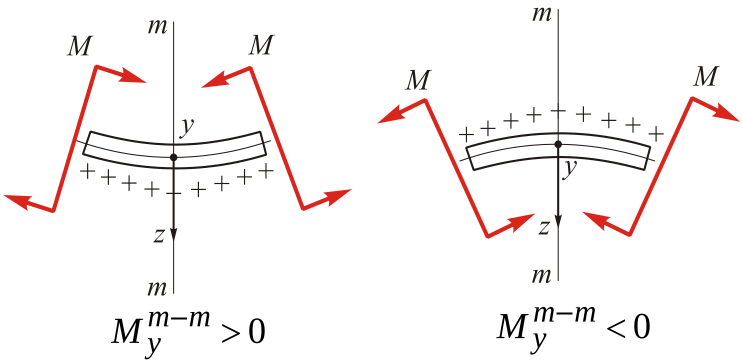


Fig. 3

The values of the moments M_0, M_2 are the following:

$$M_0 = -\frac{qa}{2}\times\frac{2}{3}a = -60 \text{ kNm}, \qquad M_2 = +\frac{qd^2}{2} = +10 \text{ kNm}.$$

(3) Calculating the unknown bending moment M_1 from the equation of three moments.

In general, the equation of three moments looks like:

$$M_0 l_1 + 2M_1(l_1 + l_2) + M_2 l_2 = -6EI(\alpha + \beta),$$

where l_1 and l_2 are the lengths of the left and right span respectively, M_0 is internal moment in cross-section of left support, M_1 – unknown internal moment in cross-section of middle support, M_2 is internal moment in cross-section of right support.

We have already defined the values of M_0 and M_2. For our case, l_1 and l_2 are correspondingly the lengths of left and right spans which are equal to $b = 2$ m and $c = 2$ m. The angles α, β are really the slopes which are generated by only external forces and moments applied correspondingly to the left and right span: α – angle in right support of left span and β – in left support of right span. ***Note, that external forces and moments which were earlier included into M_0, M_2 calculating, should***

not be included into α, β calculating. Left and right spans are shown on Figs 4 and 5 with corresponding shapes of deflected curve under loading mentioned above. Due to the M external moment is applied in midsection, it may be considered as the deflection generator of left or right span, depending on our wish. It this solution, we will assume the M moment be applied to left span. ***Note also, that the angles corresponding to convex deflection are assumed to be positive in three moment equation and vice versa.***

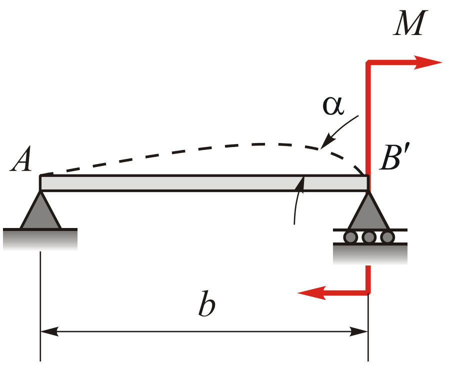


Fig. 4

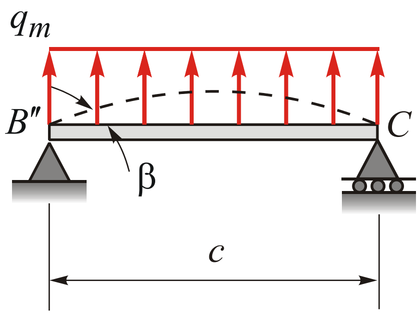


Fig. 5

(a) Let us define the α and β angles using well-known formula from teaching aids:

$$\alpha < 0 \qquad \alpha = -\frac{Mb}{3EI} = -\frac{20}{3EI} \left[\frac{\text{kNm}^2}{EI}\right].$$

Note, that $\alpha < 0$ due to concave shape of left span deflection curve which is assumed to be negative in proving three moment equation. Due to this assumption β angle will also be negative:

$$\beta < 0 \qquad \beta = -\frac{qc^3}{24EI} = -\frac{20}{3EI} \left[\frac{\text{kNm}^2}{EI}\right].$$

(b) Let us define the α and β angles using Mohr's method. For this purpose, we will consider the left and right spans under external loadings as the force systems (F) and will design two corresponding unit systems applying unit dimensionless moment $\overline{M} = 1$ in right support of the left span (to calculate α angle) and unit dimensionless moment $\overline{M} = 1$ in left support of the right span (to calculate β angle).

Note, that unit moments are applied in arbitrary directions and results of calculation may be positive or negative depending on the $\overline{\boldsymbol{M}}=\mathbf{1}$ ***direction.***

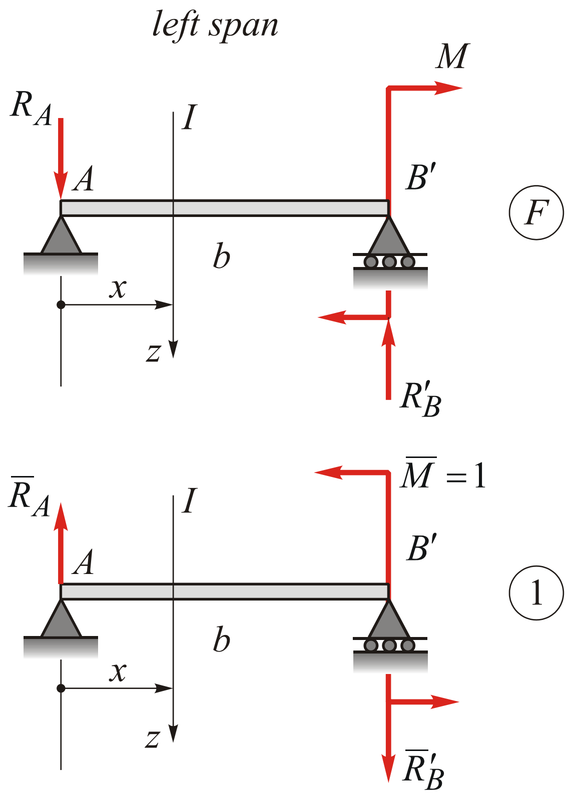


Fig. 6

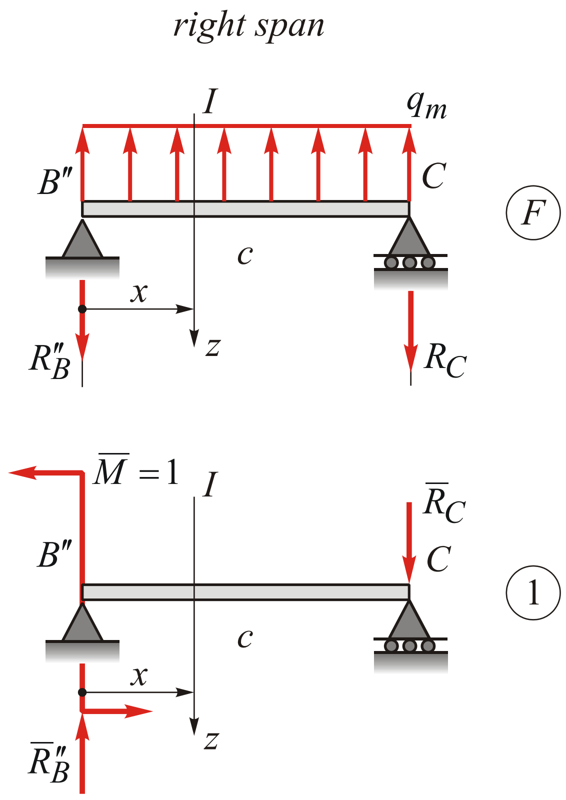


Fig. 7

Calculating the reactions in the unit systems **(clockwise rotation is assumed to be positive)**:

left span $\sum M_{B'}=0=-\overline{R}_A b+\overline{M} \;\rightarrow\; \overline{R}_A=1/b=0.5\,\text{m}, \quad \overline{R}_{B'}=\overline{R}_A=0.5\,\text{m}$,

right span $\sum M_{B''}=0=\overline{R}_C c-\overline{M} \;\rightarrow\; \overline{R}_C=1/c=0.5\,\text{m}, \quad \overline{R}_{B''}=\overline{R}_C=0.5\,\text{m}$.

Calculating the reactions in the force systems:

left span $\sum M_{B'}=0=-R_A b+M \;\rightarrow\; R_A=M/b=10/2=+5\,\text{kN}, \quad R_{B'}=+5\,\text{kN}$,

right span $\sum M_{B''}=0=R_C c-q_m c^2/2 \;\rightarrow\; R_C=+20\,\text{kN}, \quad R_{B''}=R_C=+20\,\text{kN}$.

Equations of bending moments are the following:

left span $M_{yF}^{I}(x)=-R_A x=-5x$,

$$\overline{M}_y^{I}(x)=\overline{R}_A x=0.5x.$$

right span $M_{yF}^{I}(x)=qx^2/2-R''_B x=(10x^2-20x)$,

$$\overline{M}_y^{I}(x)=-\overline{M}+\overline{R}''_B x=(-1+0.5x).$$

Results of the Mohr's method calculations are:

$$\alpha = \frac{1}{EI}\left[\int_0^b (-5x)(0.5x)\,dx\right] = \frac{1}{EI}\left[-2.5\frac{b^3}{3}\right] = -\frac{20}{3EI},$$

$$\beta = \frac{1}{EI}\left[\int_0^c \left(10x^2 - 20x\right)(-1+0.5x)\,dx\right] = +\frac{20}{3EI}.$$

Taking into account the notations mentioned above, negative values of these angles, i.e. $\alpha = -\frac{20}{3EI}$ and $\beta = -\frac{20}{3EI}$ are substituted into three moment equation:

$$(-60)\times 2 + 2M_1(2+2) + (+10)\times 2 = -6EI\left(-\frac{20}{3EI} - \frac{20}{3EI}\right).$$

After substituting we obtain the result: $M_1 = +22.5\,\text{kNm}$. It means that static indeterminacy of specified beam is opened and it is possible to determine the internal forces in equivalent system shown on Fig. 1. ***Note, that in the case of negative M_1 value it should be applied to both spans of equivalent system in opposite directions.***

(4) Considering the left and right spans separately and constructing the internal force factors diagrams for each of them. The spans are shown on Figs. 8 and 9.

(a) left span (see Fig. 8). ***Note, that clockwise rotation is assumed to be positive in the reactions calculating.***

$$\sum M_{B'} = 0: \quad -\frac{q_m a}{2}\left(\frac{2}{3}a + b\right) + R_{A_{orig}} b + M - M_1 = 0.$$

$$R_{A_{orig}} = \frac{1}{b}\left(M_1 - M + \frac{q_m a}{2}\left(\frac{2}{3}a + b\right)\right) = \frac{1}{2}\left(22.5 - 10 + \frac{30\times 2}{2}\left(\frac{2}{3}\times 2 + 2\right)\right) = +66.25\,\text{kN}$$

(actual direction upwards).

$$\sum M_A = 0: \quad -\frac{q_m a^2}{3} + R'_{B_{orig}} b + M - M_1 = 0.$$

$$R_{B'_{orig}} = \frac{1}{b}\left(M_1 - M + \frac{q_m a^2}{3}\right) = \frac{1}{2}\left(22.5 - 10 + \frac{30\times 4}{2}\right) = +36.25\,\text{kN}$$

(actual direction downwards).

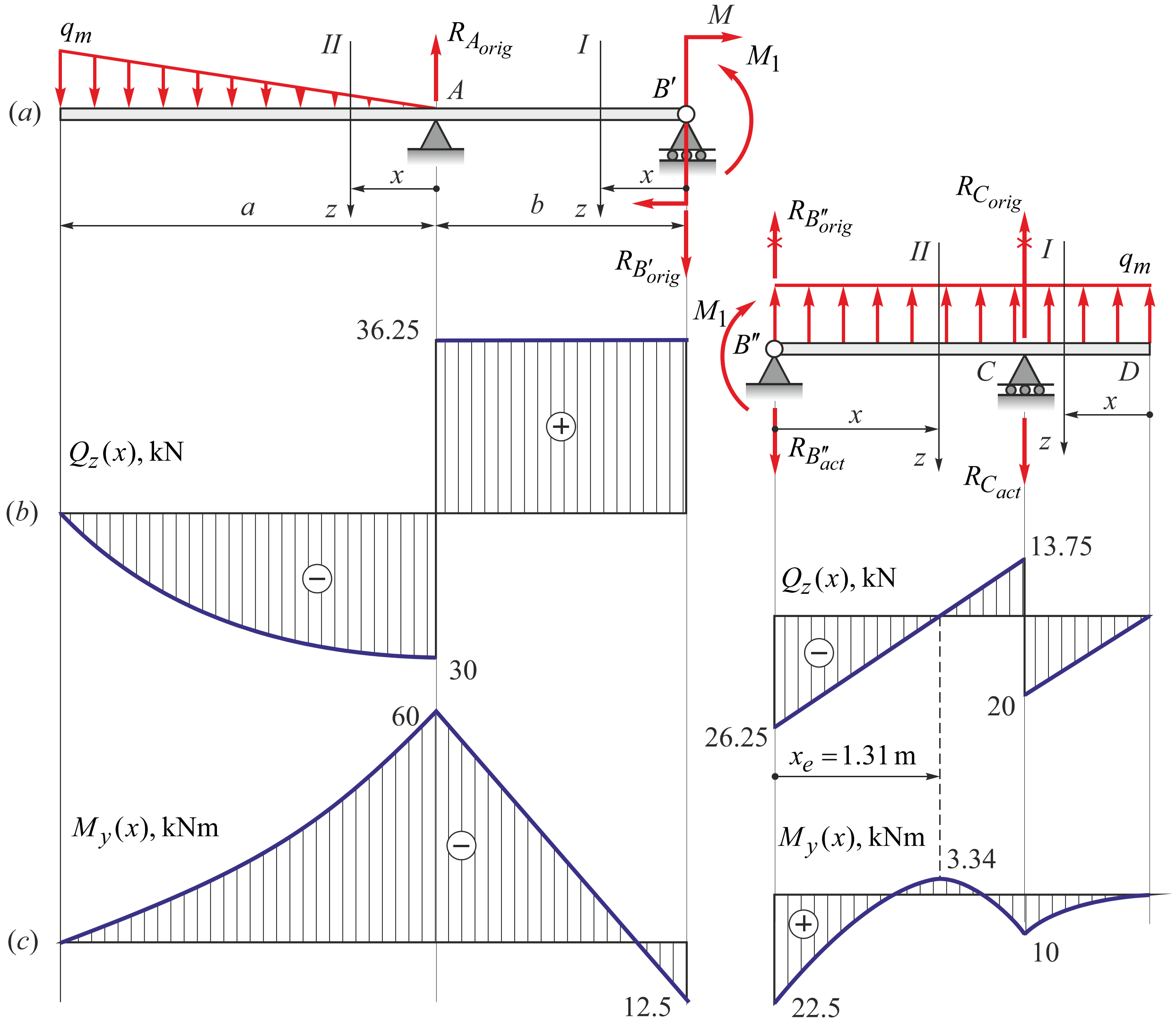


Fig. 8 Fig. 9

Checking:

$$\sum F_z = 0: \quad +\frac{q_m a}{2} - R_{A_{orig}} + R'_{B_{orig}} = 0 \quad \rightarrow \quad +\frac{20 \times 3}{2} - 66.25 + 36.25 = 0.$$

Equations of internal forces are the following:

$I - I: \quad 0 \le x \le b$

$$Q_z^I(x) = R'_{B_{orig}} = +36.25\ \text{kN},$$

$$M_y^I(x) = -R'_{B_{orig}}\, x - M + M_1 \,|_{x=0} = 12.5 \,|_{x=2} = -60\ \text{kNm}.$$

$II - II: \quad 0 \le x \le a$

$$Q_z^{II}(x) = R'_{B_{orig}} - R_{A_{orig}} + \frac{q_m x^2}{2a} \,|_{x=0} = -30 \,|_{x=a} = 0,$$

$$M_y^{II}(x) = -R'_{B_{orig}}\,(x+b) - M + M_1 + R_{A_{orig}}\, x - \frac{q x^3}{6a} \,|_{x=0} = -60 \,|_{x=a} = 0.$$

(b) right span (see Fig. 9). ***Note, that clockwise rotation is assumed to be positive in the reactions calculating.***

$$\sum M_c = 0: \quad -\frac{q_m d^2}{2} + \frac{q_m c^2}{2} + R''_{B_{orig}} c + M_1 = 0.$$

$$R''_{B_{orig}} = \frac{1}{c}\left(+ \frac{q_m d^2}{2} - \frac{q_m c^2}{2} - M_1 \right) = \frac{1}{2}\left(\frac{20 \times 1}{2} - \frac{20 \times 4}{2} - 22.5 \right) = -26.25\,\text{kN}$$

(actual direction downwards).

$$\sum M_{B''} = 0: \quad -R_{C_{orig}} c + M_1 - \frac{q_m}{2}(c+d)^2 = 0.$$

$$R_{C_{orig}} = \frac{1}{c}\left(M_1 - \frac{q_m}{2}(c+d)^2 \right) = \frac{1}{2}\left(22.5 - \frac{20}{2}(2+1)^2 \right) = -33.75\,\text{kN}$$

(actual direction downwards).

Checking:

$$\sum F_z = 0: \; -q(c+d) + R_{C_{act}} + R''_{B_{act}} = 0 \;\; \rightarrow \;\; -60 + 33.75 + 26.25 = 0.$$

Equations of internal forces are the following:

$I - I: \quad 0 \le x \le d$

$$Q_z^I(x) = -q_m x\,|_{x=0} = 0\,|_{x=d} = -20(\text{kN}),$$

$$M_y^I(x) = \frac{q_m x^2}{2}\,|_{x=0} = 0\,|_{x=d} = 10(\text{kNm}).$$

$II - II: \quad 0 \le x \le c$

$$Q_z^{II}(x) = -R''_{B_{act}} + q_m x\,|_{x=0} = -26.25\,|_{x=c} = 13.75\,\text{kN},$$

$$M_y^{II}(x) = M_1 - R''_{B_{act}} x + \frac{q_m x^2}{2}\,|_{x=0} = 22.5\,|_{x=c} = 10\,\text{kNm}.$$

Extremal bending moment calculating:

$$Q_z^{II}(x_e) = -R''_{B_{act}} + q x_e = 0 \;\; \rightarrow \;\; x_e = \frac{R''_{B_{act}}}{q_m} = \frac{26.25}{20} = 1.31\,\text{m}.$$

$$M_y^{II}(x_e) = M_{y_{\max}}^{II} = M_1 - R''_{B_{act}} x_e + \frac{q_m x_e^2}{2} = 10 - 26.25 \times 1.31 + 20 \times 1.31^2 / 2 = -3.34(\text{kNm}).$$

By connecting the graphs of internal forces for two spans we get the solution of the problem shown on Fig. 10.

(B) Solution by the force method.

First of all let us choose the base system (*BS*) and design corresponding equivalent system (*ES*) (see Fig. 11). Designing correspondent equivalent system is also shown on Fig. 11. The effect of middle support is replaced in equivalent system by unknown reaction (force) X_1. Its value must be found using the equation of deflection compatibility, which is represented as **canonical equation of the force method**. Their geometrical sense is in total zero vertical deflection of vertically immobile *B* point of equivalent system. This deflection is really a geometric sum of the *B*-point deflection generated by external forces and, secondly, by unknown X_1 force. This canonical equation has the shape:

$$\delta_{vert_{p.B}}(X_1, F) = 0 \quad \text{or} \quad \delta_{11}X_1 + \Delta_{IF} = 0.$$

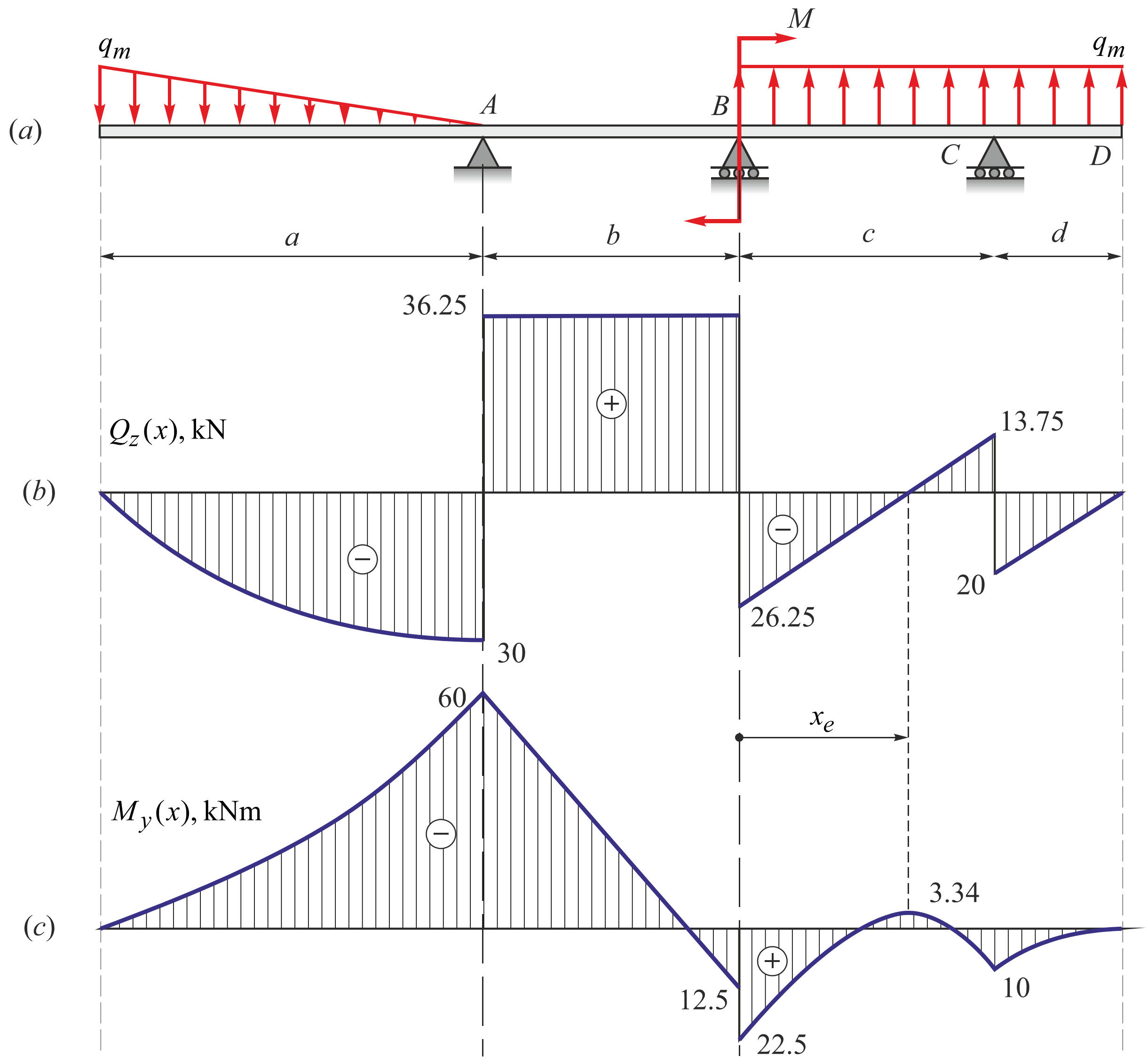


Fig. 10

To find two coefficients δ_{11} and Δ_{1F} it is necessary to design the force (F) and unit (1) systems. They are shown on Fig. 11. ***Note, that the force system is the base system with only external forces applied***. Unit system is the base system with unit $\overline{X}_1$ force applied. They are shown on Fig. 11.

The unknown reactions R_A and R_C in the force system (F) we will calculate using the equations of statics:

$$\sum M_A = 0 = -\frac{q_m a}{2} \times \frac{2}{3} a + M + R_C (b+c) - q_m (c+d) \left(\frac{c+d}{2} + b \right).$$

$$R_C = \frac{1}{b+c} \left(\frac{q_m a^2}{3} - M + q_m (c+d) \left(\frac{c+d}{2} + b \right) \right) = \frac{1}{4} \left(\frac{20 \times 9}{3} - 10 + 20 \times 3 \left(\frac{3}{2} + 2 \right) \right) = +65 \text{ kN}$$

(actual direction downwards).

$$\sum M_C = 0 = -\frac{q_m a}{2} \times \left(\frac{2}{3} a + b + c \right) + M + R_A (b+c) + \frac{q_m c^2}{2} - \frac{q_m d^2}{2}.$$

$$R_A = \frac{1}{b+c} \left(\frac{q_m a}{2} \left(\frac{2}{3} a + b + c \right) - M - \frac{q_m c^2}{2} + \frac{q_m d^2}{2} \right) =$$
$$= \frac{1}{4} \left(\frac{20 \times 9}{2} \left(\frac{2}{3} \times 3 + 2 + 2 \right) - 10 - \frac{20 \times 4}{2} + \frac{20 \times 1}{2} \right) = +35 \text{ kN}$$

(actual direction upwards).

Checking:

$$\sum F_z = 0 = \frac{q_m a}{2} - q_m (c+d) - R_A + R_C = \frac{20 \times 3}{2} - 20(2+1) - 35 + 65 = 0.$$

The unknown reactions $\overline{R}_A$ and $\overline{R}_c$ in the unit system (1) we will also calculate using the equations of statics:

$$\sum M_A = 0 = +\overline{R}_C (b+c) - \overline{X}_1 b = 0 \;\rightarrow\; \overline{R_C} = \frac{1}{2} (\text{dimensionless})$$

(actual direction upwards).

$$\sum M_C = 0 = -\overline{R}_A (b+c) + \overline{X}_1 c = 0 \;\rightarrow\; \overline{R_A} = \frac{1}{2} (\text{dimensionless})$$

(actual direction upwards).

Checking: $\sum F_z = 0 = \bar{R}_A + \bar{R}_C - \bar{X}_1 = \frac{1}{2} + \frac{1}{2} - 1 = 0.$

Equations of internal forces in the force and unit systems are the following:

$I - I: \;\; 0 \le x \le d$

$$M_{yF}^{I}(x) = \frac{q_m x^2}{2} = \left(10x^2\right),$$

$$\bar{M}_y^{I}(x) = 0.$$

$II - II: \;\; 0 \le x \le c$

$$M_{yF}^{II}(x) = q_m d\left(\frac{d}{2} + x\right) + \frac{q_m x^2}{2} - R_C x = \left(10 - 45x + 10x^2\right),$$

$$\bar{M}_y^{II}(x) = \left(-\frac{x}{2}\right).$$

$III - III: \;\; 0 \le x \le b$

$$M_{yF}^{III}(x) = q_m d\left(\frac{d}{2} + c + x\right) + q_m c\left(\frac{c}{2} + x\right) - R_C(c + x) - M = (-50 - 5x),$$

$$\bar{M}_y^{III}(x) = \left(-1 + \frac{x}{2}\right).$$

$IV - IV: \;\; 0 \le x \le a$

$$M_{yF}^{IV} = q_m d\left(\frac{d}{2} + c + b + x\right) + q_m c\left(\frac{c}{2} + b + x\right) - R_C(c + b + x) - M + R_A x + \frac{q_m x^3}{6a} =$$
$$= \left(\frac{10}{9}x^3 + 30x - 80\right),$$

$$\bar{M}_y^{IV}(x) = 0.$$

Calculating the canonical equation coefficients applying Mohr's method.

$$\Delta_{IF} = \frac{1}{EI}\left[\int_0^1 \left(10x^2\right)(0)\,dx + \int_0^2 \left(10 - 45x + 10x^2\right)\left(-\frac{x}{2}\right) + \int_0^2 (-50 - 5x)\left(-1 + \frac{x}{2}\right)dx + \right.$$
$$\left. + \int_0^3 \left(\frac{10}{9}x^3 + 30x - 80\right)(0)\,dx\right] = +\frac{250}{3EI}.$$

$$\delta_{11} = \frac{1}{EI}\left[\int_0^1 (0)(0)\,dx + \int_0^2 \left(-\frac{x}{2}\right)\left(-\frac{x}{2}\right)dx + \int_0^2 \left(-1 + \frac{x}{2}\right)\left(-1 + \frac{x}{2}\right)dx + \int_0^3 (0)(0)\,dx\right] = +\frac{4}{3EI}.$$

Note, that δ_{11} coefficient may be also calculated applying graphical method.

Substituting these coefficients into canonical equation is the following:

$$\frac{4}{3EI}\times X_1+\frac{250}{3EI}=0 \quad \text{and} \quad X_1=-62.5\ \text{kN}.$$

Note, that "minus sign" means that actual direction of X_1 force is opposite to its upwards original direction in equivalent system (see Fig. 11c).

After X_1 finding the equivalent system becomes available for shear forces and bending moments calculating. Let us preliminary determine the reactions R_A^* and R_C^* in equivalent system. Their original directions are shown on Fig. 11. Note, that original direction of X_1 in equivalent system should be changed on opposite before these calculating.

$$\sum M_A=0=+X_1b+M-q_m\left(c+d\right)\left(\frac{c+d}{2}+b\right)-R^*_{C_{orig}}\left(b+c\right)-\frac{q_ma}{2}\times\frac{2a}{3}\ .$$

$$R^*_{C_{orig}}=\frac{1}{b+c}=\left(X_1b+M-q_m\left(c+d\right)\left(\frac{c+d}{2}+b\right)-\frac{q_ma}{2}\times\frac{2a}{3}\right)=-33.75\ \text{kN}$$

(actual direction downwards).

$$\sum M_C=0=-X_1c+M+R^*_{A_{orig}}\left(b+c\right)-\frac{q_ma}{2}\left(\frac{2a}{3}+b+c\right)+\frac{q_mc^2}{2}-\frac{q_md^2}{2}.$$

$$R^*_{A_{orig}}=\frac{1}{b+c}\left(X_1c-M+\frac{q_ma}{2}\left(\frac{2a}{3}+b+c\right)-\frac{q_mc^2}{2}+\frac{q_md^2}{2}\right)=+66.25\,\text{kN}$$

(actual direction upwards).

Checking:

$$\sum F_z=0=X_1+R^*_{C_{act}}+\frac{qa}{2}-R^*_{A_{orig}}-q\left(c+d\right)=62.5+33.75+30-66.25-60\equiv 0\,.$$

Equations of internal forces in equivalent system are the following:

$I-I:\ \ 0<x<d$

$$Q_z^I\left(x\right)=-q_mx=-20x\big|_{x=0}=0\big|_{x=1}=-20\ \text{kN},$$

$$M_y^I(x)=\frac{q_m x^2}{2}=10x^2\Big|_{x=0}=0\Big|_{x=1}=+10\text{ kNm.}$$

$II-II:\quad 0<x<c$

$$Q_z^{II}(x)=-q_m(x+d)+R^*_{C_{act}}=-20(x+1)+33.75=$$

$$=(-20x+13.75)\Big|_{x=0}=+13.75\Big|_{x=2}=-26.25\text{ kN,}$$

$$M_y^{II}(x)=+q_m(d+x)\frac{(d+x)}{2}-R^*_{C_{act}}x=$$

$$=\left(10x^2-13.75x+10\right)\Big|_{x=0}=+10\Big|_{x=2}=+22.5\text{ kNm.}$$

Due to $Q_z^{II}(x)$ function changes its sign from "plus" to "minus" it is necessary to find $M_y^{II}(x)$ extremal value:

(a) extremum coordinate finding by equating to zero $Q_z^{II}(x)$ function:

$$Q_z^{II}(x_e)=(-20x_e+13.75)=0\ \rightarrow\ x_e=+\frac{13.75}{20}=+0.69\text{ m.}$$

(b) calculating $M^{II}_{y\max}$ value substituting x_e value into $M_y^{II}(x)$ equation:

$$M_y^{II}(x_e)=M^{II}_{y_{\max}}=\left(10x_e{}^2-13.75x_e+10\right)=-3.34\text{ kNm.}$$

$III-III:\quad 0<x<b$

$$Q_z^{III}(x)=R^*_{C_{act}}-q_m(c+d)+X_1=+36.25\text{ kN,}$$

$$M_y^{III}(x)=-R^*_{C_{act}}(c+x)-M-X_1x+q_m(c+d)\left(\frac{c+d}{2}+x\right)=$$

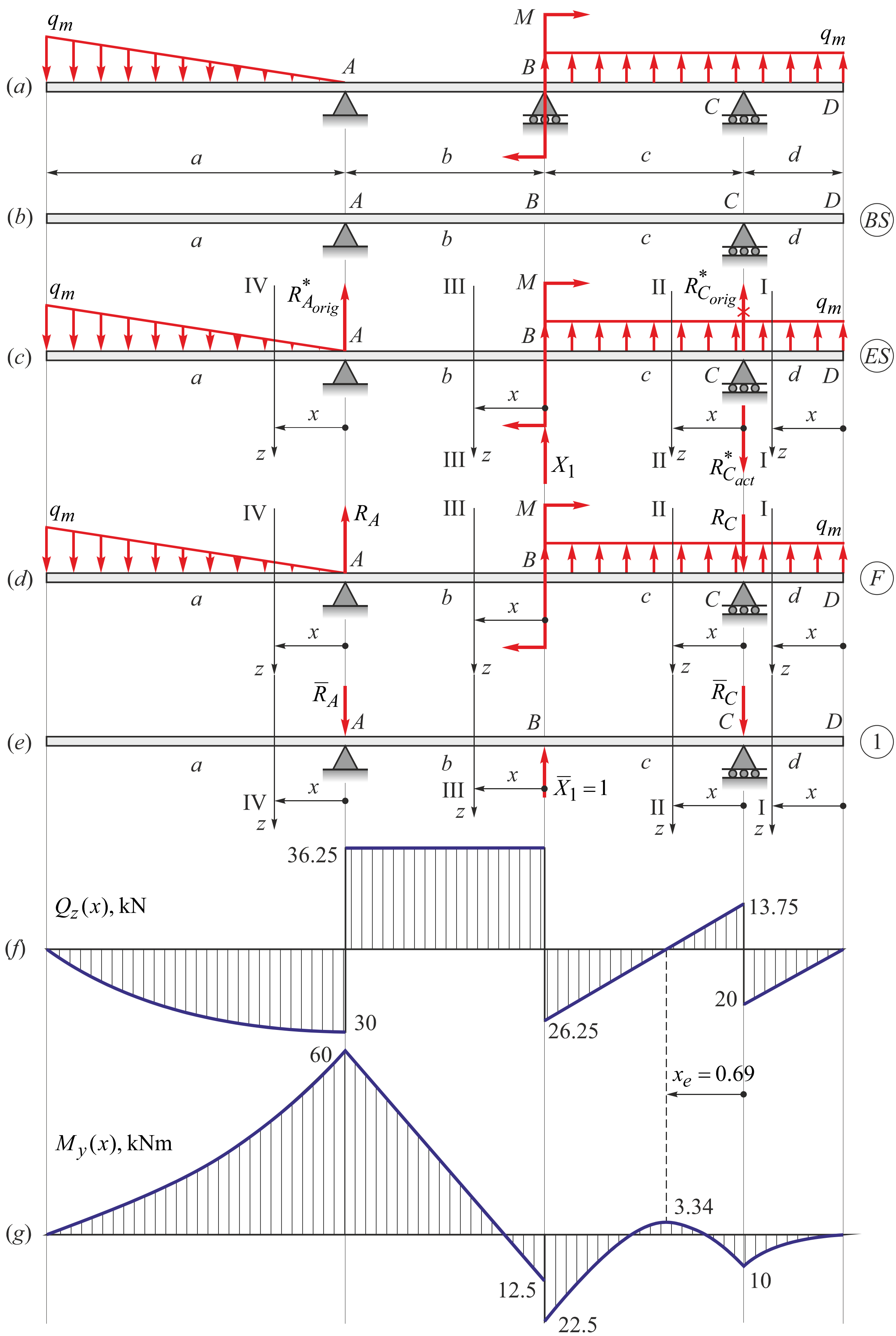
$$=(12.5-36.25x)\Big|_{x=0}=12.5\Big|_{x=2}=-60\text{ kNm.}$$

$IV-IV:\quad 0<x<a$

$$Q_z^{IV}(x)=R^*_{C_{act}}-q_m(c+d)+X_1+\frac{q_m x^2}{2a}-R^*_{A_{orig}}=\left(\frac{10}{3}x^2-30\right)\Bigg|_{x=0}=-30\Big|_{x=3}=0\text{ kN,}$$

$$M_y^{IV}(x)=-R^*_{C_{act}}(c+b+x)+q_m(c+d)\left(\frac{c+d}{2}+b+x\right)-M-$$

$$-X_1(b+x)+R^*_{A_{orig}}x-\frac{q_m x^3}{6a}=\left(-\frac{10}{9}x^3+30x-60\right)\Bigg|_{x=0}=-60\Big|_{x=3}=0\text{ kNm.}$$



(a)
(b)
(c)
(d)
(e)
(f)
(g)
BS
ES
F
1
36.25
Qz(x), kN
13.75
30
26.25
20
60
xe = 0.69
My(x), kNm
3.34
12.5
22.5
10

Fig. 11

The $Q_z(x)$ and $M_y(x)$ graphs are represented on Fig. 11 (f, g).

General conclusion. Due to X_1 force is really the reaction in middle support, it may be compared with the “abrupt” on the shear force graph, designed in result of first solution applying three moment equation. This “abrupt” is equal to ($36.25 + 26.25 = 62.5$ kN). It’s coincidence with the value of X_1 force supports the accuracy of this problem solution.

The “abrupt” on the $M_y(x)$ graph in *B*-point is equal to external *M* value 10 kNm and internal moment in *B* section (equal to 22.5 kNm) is really unknown M_1 moment which has been found earlier in three moment equation.

Totally, the graphs of $Q_z(x)$ and $M_y(x)$ shown on the Figs. 10 and 11 are identical.