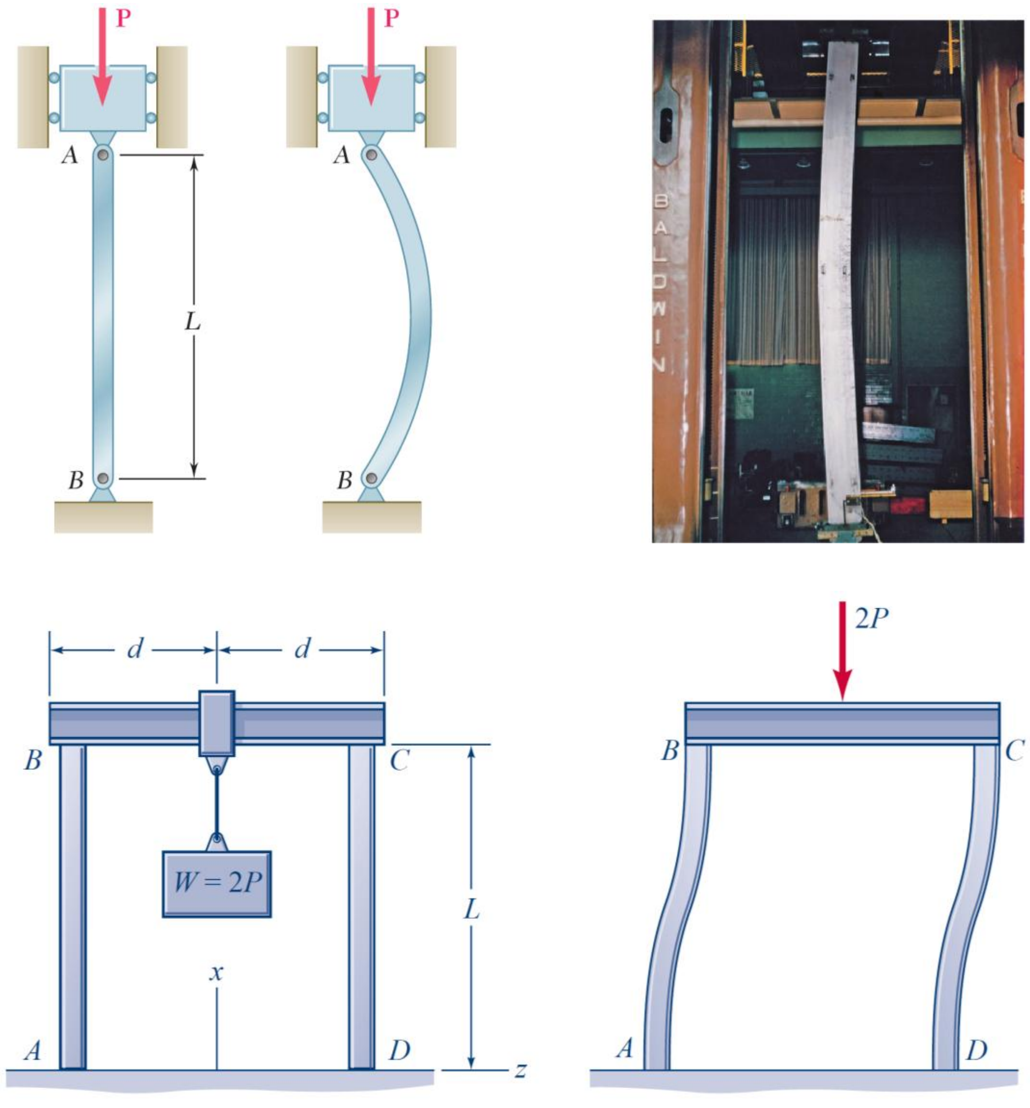
LECTURE 25 Buckling of Columns (Part 1)
Short version

1 Stability of Equilibrium

By stability is meant the property of a system to recover its original state after it has been displaced from the position of equilibrium. If a system does not possess this property, it is classified as **unstable** (see Fig. 1).



(*a*) Unbuckled configuration. (*b*) Buckled configuration.

Fig. 1

An ideal system is given a displacement from the position of equilibrium. If after removing the causes of the displacement the system returns to its initial state of equilibrium, the latter is considered **stable**. Otherwise it is **unstable** (see Fig. 2).

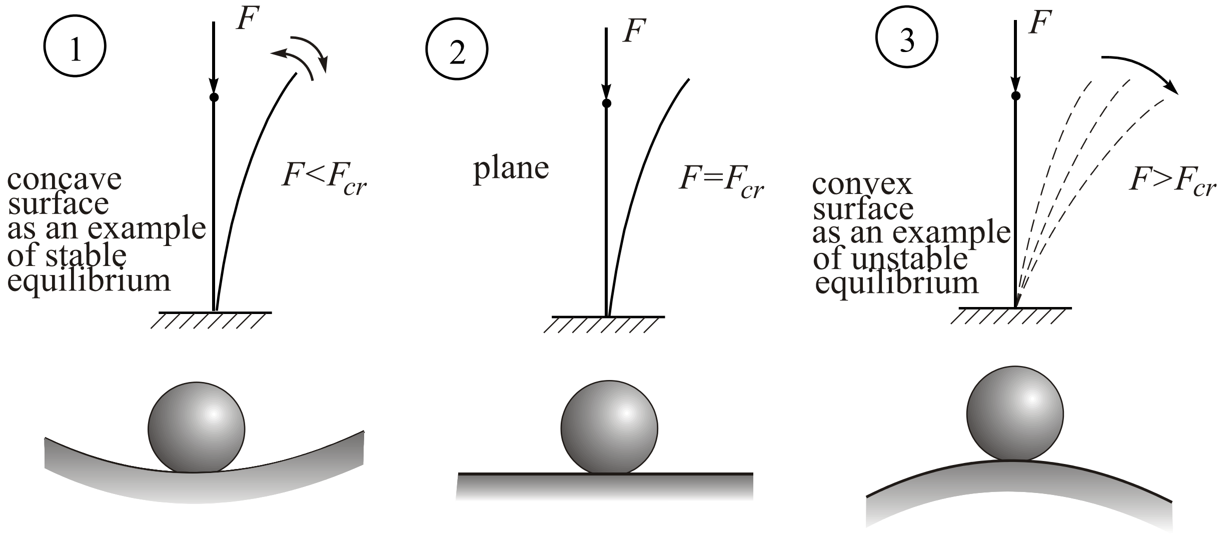


Fig. 2

For the great majority of elastic systems such an approach to the analysis of stability (**buckling**) makes it possible to determine the values of external forces at which the stable position of equilibrium becomes unstable. Such forces are called **critical loads** and are regarded as limiting for a structure.

It is evident, that

$$F_{cr} < \sigma_y A, \tag{1}$$

where σ_y is the **yield limit**, A is the area of the bar. In buckling calculations the working load is assigned as the n-th fraction of the critical load. The quantity n is the **stability factor of safety**. Maximum working value of compressive force may be calculated as

$$F = \frac{F_{cr}}{n}. \tag{2}$$

2 The Euler's Problem

We shall begin the study of buckling of elastic systems with the simplest problem of equilibrium of a bar compressed by central forces F.

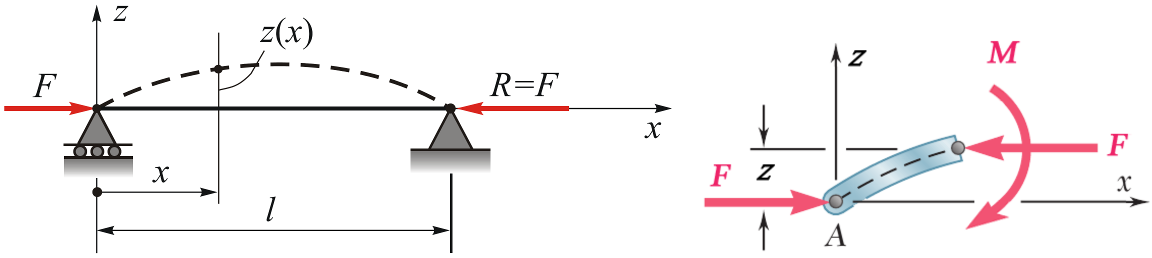


Fig. 3

Suppose a compressed bar is deflected slightly for some reason. Consider the conditions under which an equilibrium of the bar with the deflected axis is possible.

The co-ordinates of points of the elastic curve of the bar are denoted by x and z. For small deflections, as you know, Euler's equation connects bending moments and deflections:

$$EIz''(x) = -M_y(x) = -Fz(x), \quad \sigma \le \sigma_{pr}, \tag{3}$$

where σ_{pr} – **proportional limit**.

The **bending of the bar occurs in the plane of minimum rigidity, and so the quantity I is understood to be the minimum moment of interia of the section**:

$$EI_{\min} z''(x) = -Fz(x),$$

$$z''(x) = -\frac{F}{EI_{\min}} z(x). \tag{4}$$

Let's denote the factor

$$\frac{F}{EI_{\min}} = k^2. \tag{5}$$

Equation (5) then becomes

$$z''(x) + k^2 z(x) = 0, \tag{6}$$

whence solution will be:

$$z(x) = C_1 \sin kx + C_2 \cos kx.$$

The constants C_1 and C_2 must be chosen so as to satisfy the boundary conditions: when $x = 0$, $z = 0$ and when $x = l$, $z = l$. From the first condition it follows that $C_2 = 0$, and from the second condition

$$C_1 \sin kl = 0. \tag{7}$$

This equation has two possible solutions: either $C_1 = 0$ or $\sin kl = 0$. In the first case the displacements z are identically zero for $C_1 = C_2 = 0$, and so the bar maintains the straight-line configuration. This case is of no interest.

In the second case

$$kl = n\pi, \tag{8}$$

where n is arbitrary integer.

Taking into account expression (5), we obtain

$$F = \frac{\pi^2 n^2}{l^2} EI_{\min}. \tag{9}$$

This means that, in order for the bar to maintain a curvilinear configuration, the force F must take a definite value. This minimum force F is when $n = 1$:

$$F_{cr} = \frac{\pi^2 EI_{\min}}{l^2}. \tag{10}$$

This force is termed the **first critical force** or **Euler's buckling load**. When $n = 1$

$$kl = \pi \tag{11}$$

and the elastic curve equation becomes

$$z(x) = C_1 \sin \frac{\pi x}{l}. \tag{12}$$

The bar bends a half-wave of a sine curve with a maximum deflection C_1.

3 Effect of Boundary Conditions on the Critical Load

In the range of small displacements a pin-ended bar buckles in a half-wave of a sine curve and the critical load is given by

$$F_{cr} = \frac{\pi^2 EI_{\min}}{l_0^{\,2}}, \tag{13}$$

where l_0 is the length of a **half-wave of a sine curve** ($l = l_0$, see Fig. 4).

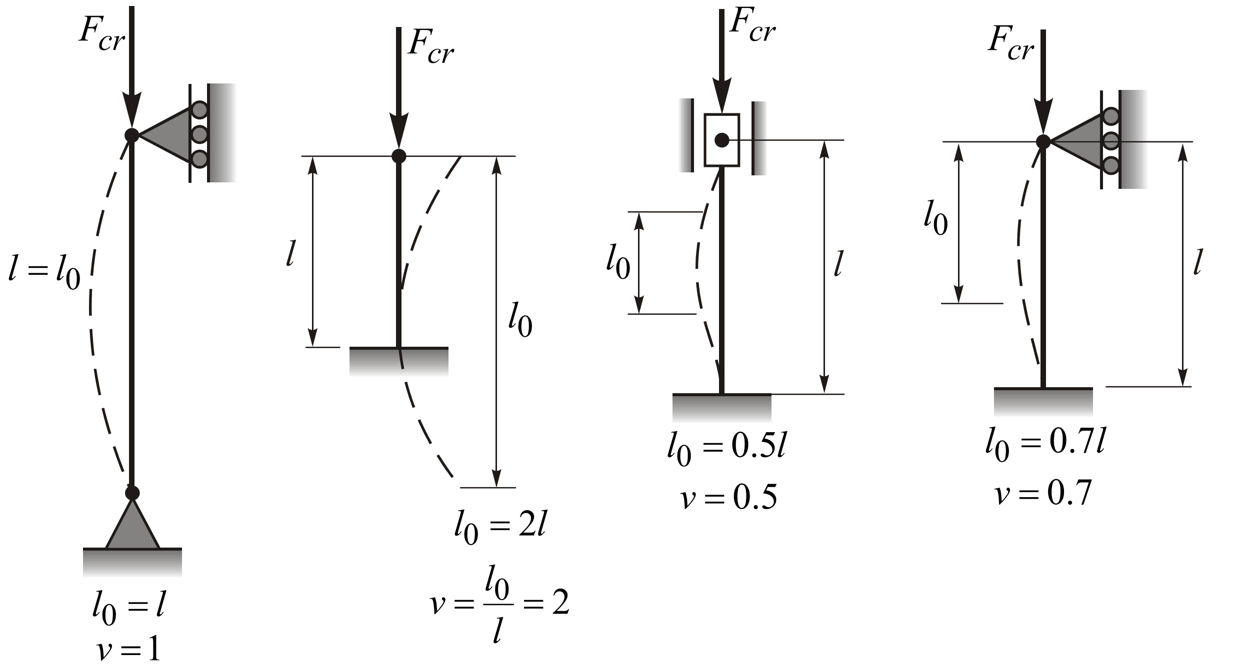


Fig. 4

If, for example, a bar is fixed at one end and free at the other, then $l_0 = 2l$.

By generalizing the above formulas we can write the general expression for the critical load of a compressed bar

$$F_{cr} = \frac{\pi^2 EI_{\min}}{(\nu l)^2}, \tag{14}$$

where ν is the so-called **length reduction factor**, and $l_0 = \nu l$ – **effective length**.

4 Critical Stresses. Conditions of the Euler's Formula Applicability

The critical stress is

$$\sigma_{cr} = \frac{F_{cr}}{A} = \frac{\pi^2 EI_{\min}}{(\nu l)^2 A} = \frac{\pi^2 Ei_{\min}^2}{(\nu l)^2}, \tag{15}$$

where "*i*" denotes the **radius of gyration** of the section:

$$i^2 = \frac{I}{A} \quad \text{or} \quad i = \sqrt{\frac{I}{A}}. \tag{16}$$

The quantity $\frac{\nu l}{i}$ is denoted by λ:

$$\frac{\nu l}{i_{\min}} = \lambda \tag{17}$$

and is called the **actual slenderness ratio** of the bar. Expression (15) for the critical stress then becomes

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2}.$$

As can be seen, the stress σ_{cr} increases as the slenderness ratio decreases.

The Euler's formula is not applicable if the stress σ_{cr} reaches the proportionality limit σ_{pr}:

$$\sigma_{cr} \le \sigma_{pr}.$$

It means that $\frac{\pi^2 E}{\lambda^2} \le \sigma_{pr}$.

From this expression the **limiting slenderness ratio** is determined

$$\lambda_{\lim} = \sqrt{\frac{\pi^2 E}{\sigma_p}}.$$

Because

$$\sigma_{cr} \le \sigma_p$$

then

$$\lambda > \lambda_{\lim}.$$

For a slenderness ratio less than $\lambda_{\lim}$ Euler's formula is inapplicable.