LECTURE 25 Buckling of Columns (Part 1)
Elastic Column Behavior

Introduction

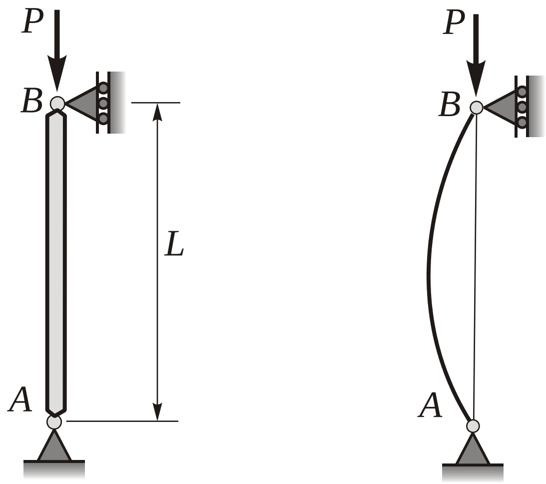
Load-carrying structures may fail in a variety of ways, depending upon the type of structure, the conditions of support, the kinds of loads, and the materials used. For instance, an axle in a vehicle may fracture suddenly from repeated cycles of loading, or a tensile member may stretch excessively, so that the structure is unable to perform its intended functions. These kinds of failures are prevented by designing structures so that the ***maximum stresses and maximum displacements (strains) remain within tolerable limits***. Thus, strength and stiffness are important factors in design.

The last type of failure is due to elastic **instability** of a structure. This type of failure is called **buckling**, which can be defined as the ***sudden, large, lateral deflection of a column owing to a small increase in an applied compressive load***. This response leads to instability and collapse of the member. The buckling phenomenon may be illustrated using wooden or metal scale with a compressive load applied at its ends. The consideration only of material stress level is not sufficient to predict the behavior of such a member.

Stability is the ability of a structure to support a given load without experiencing a sudden change in configuration. The principal difference between the theories of linear elasticity and linear stability is that, in the former, equilibrium is based upon the **undeformed geometry**, whereas in the latter, the **deformed geometry** must be considered.

We will consider specifically the **buckling of columns**, which are long, **slender structural members** loaded axially in compression (Fig. 1a). If compressed member is relatively slender, it may fail by bending or deflecting laterally (Fig. 1b) rather than by direct compression of the material. When lateral bending occurs, we say that the column has buckled. Under an increasing axial load, the lateral deflections will increase too, and eventually the column will collapse completely.

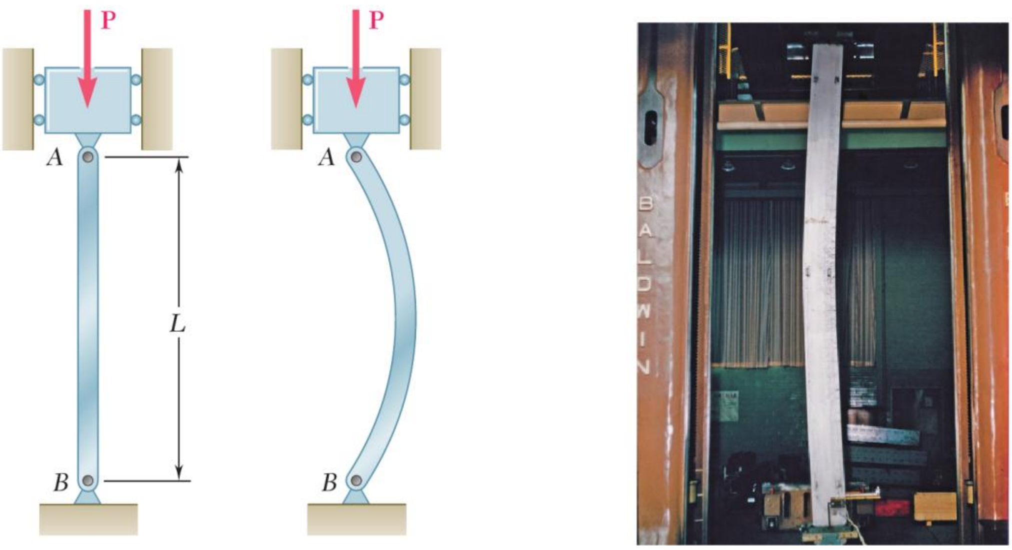
The phenomenon of buckling is not limited to columns. Buckling can occur in many kinds of structures and can take many forms. When you step on the top of an empty aluminum can, the thin cylindrical walls buckle under your weight and the can collapses. In a few bridge failures, investigators found that failure was caused by the buckling of a thin steel plate that wrinkled under compressive stresses. Buckling is also encountered in machine linkages, signposts, supports for highway overpasses, and a wide variety of other structural and machine elements. ***Buckling is one of the major causes of failures in structures, and therefore the possibility of buckling should always be considered in design***.



(a) (b)

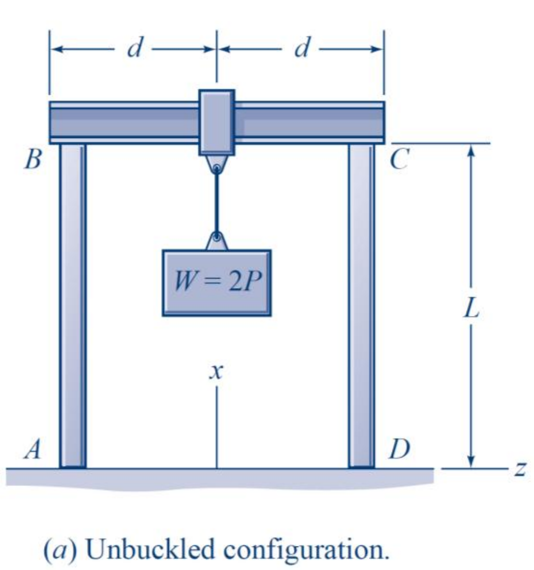
Fig. 1. Buckling of a column due to an axial compressive load *P*

The examples of buckling shown in Fig. 2. Before the buckling problem analysis, let us discuss the concept of the mechanical system stability, which is important in buckling problem description.

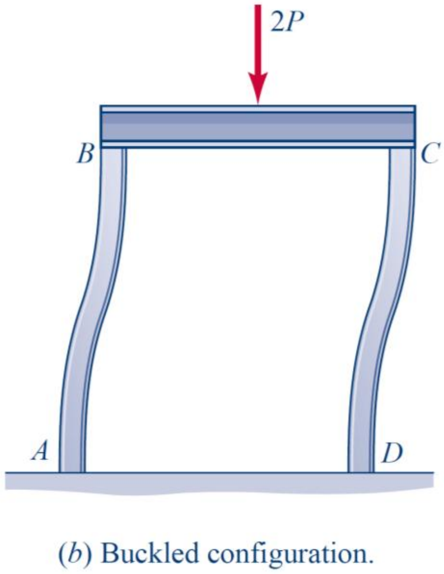


(a) (b) (c)

Fig. 2 (a) Unbuckled and (b) buckled configuration of compressed rods; (c) image of buckling tests



(a)



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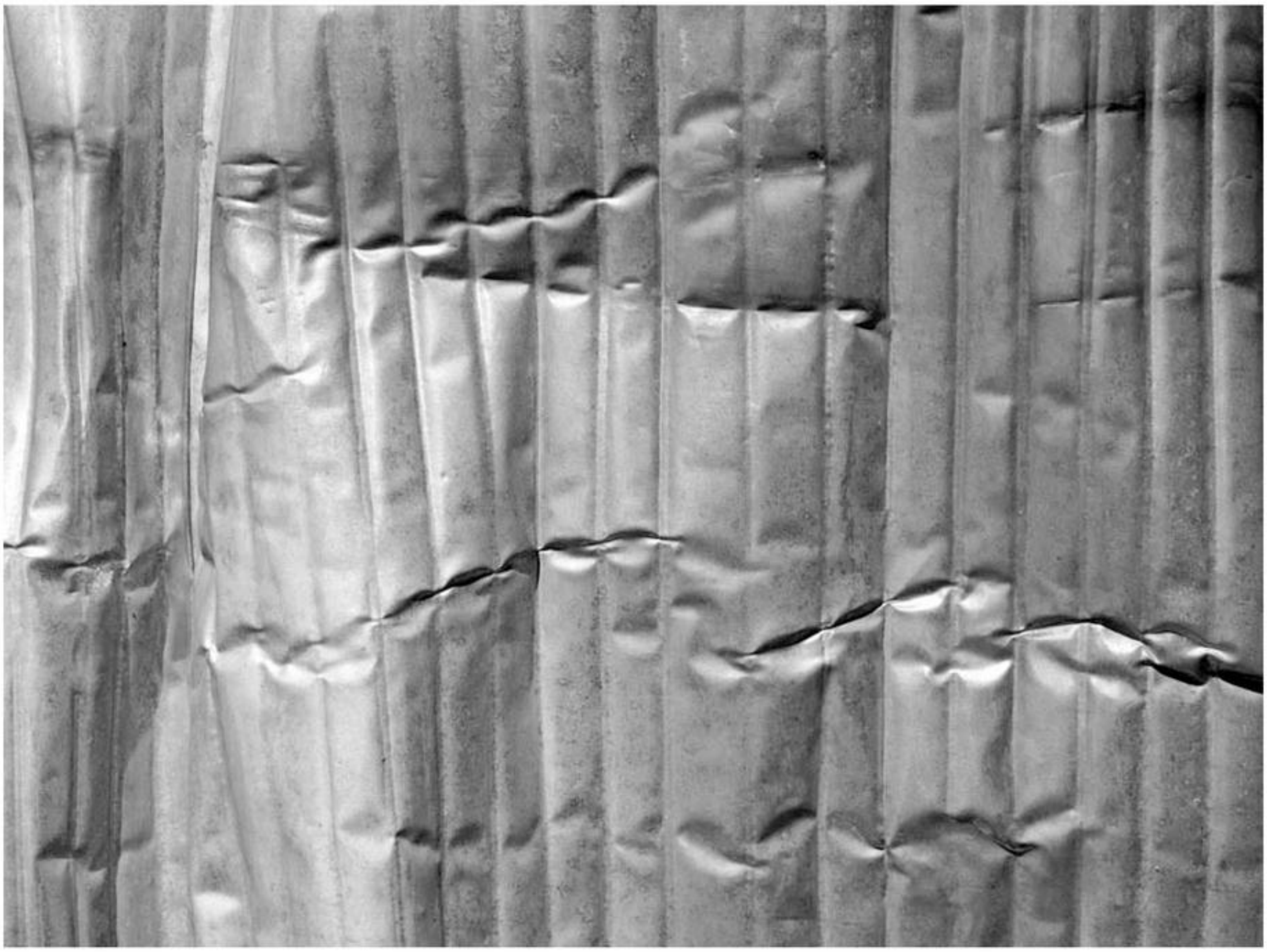
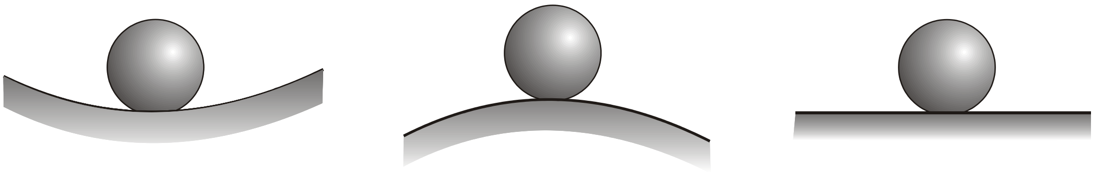


Fig. 3 (a) Unbuckled and (b) buckled configuration of compressed posts; (c) image of buckled compressed skin

An ideal mechanical system is given a displacement from the position of equilibrium. If after removing the causes of the displacement the system returns to its initial state of equilibrium, the latter is considered stable. Otherwise it is unstable. These three equilibrium conditions are analogous to those of a ball placed upon a smooth surface (Fig. 4). If the surface is concave upward, the equilibrium is stable and the ball always returns to the low point when disturbed. If the surface is convex upward, the ball can theoretically be in equilibrium on top of the surface, but the equilibrium is unstable and in reality the ball rolls away. If the surface is perfectly flat, the ball is in neutral equilibrium and remains wherever it is placed.



concave surface convex surface perfectly flat surface

Fig. 4 Ball in stable, unstable, and neutral equilibrium

In mechanical structures analysis, we shall describe the ***critical, or buckling, load, the compressive load that causes the instability***, and the **effective lengths** of columns with various restraints. The stresses associated with both elastic and inelastic buckling of columns under centric load will be considered too. Following this, we will treat eccentrically loaded columns and consider the problems of column design.

The values of external compressed forces at which the stable position of equilibrium becomes unstable. Such forces are called **critical loads** and are regarded as limiting for a structure.

It is evident, that

$$F_{cr} < \sigma_y \times A, \tag{1}$$

where σ_y is the **yield limit**, A is the area of the bar. In buckling calculations the working load is assigned as the n-th fraction of the critical load. The quantity n is the **stability factor of safety**. Maximum working (**allowable**) **value of compressive force** may be calculated as

$$F = \frac{F_{cr}}{n}. \quad (2)$$

1 Buckling of Pin-Ended Columns. The Euler's Problem

We begin our consideration of the stability behavior of columns by analyzing slender column with pinned ends (syn. **slender pin-ended column**) (Fig. 5a). The column is loaded by a vertical compressive force P that is applied through the centroid of the end cross section. The column is assumed to be perfectly straight and to be constructed of a linearly elastic material that follows Hooke's law. The column is assumed to have no **imperfections**, it is referred to as an **ideal column**.

Let us construct a coordinate system with its origin at support A and with the x axis along the longitudinal axis of the column. The z axis is directed to the left in the figure. We assume that the xz plane is a plane of symmetry of the column (Fig. 5b).

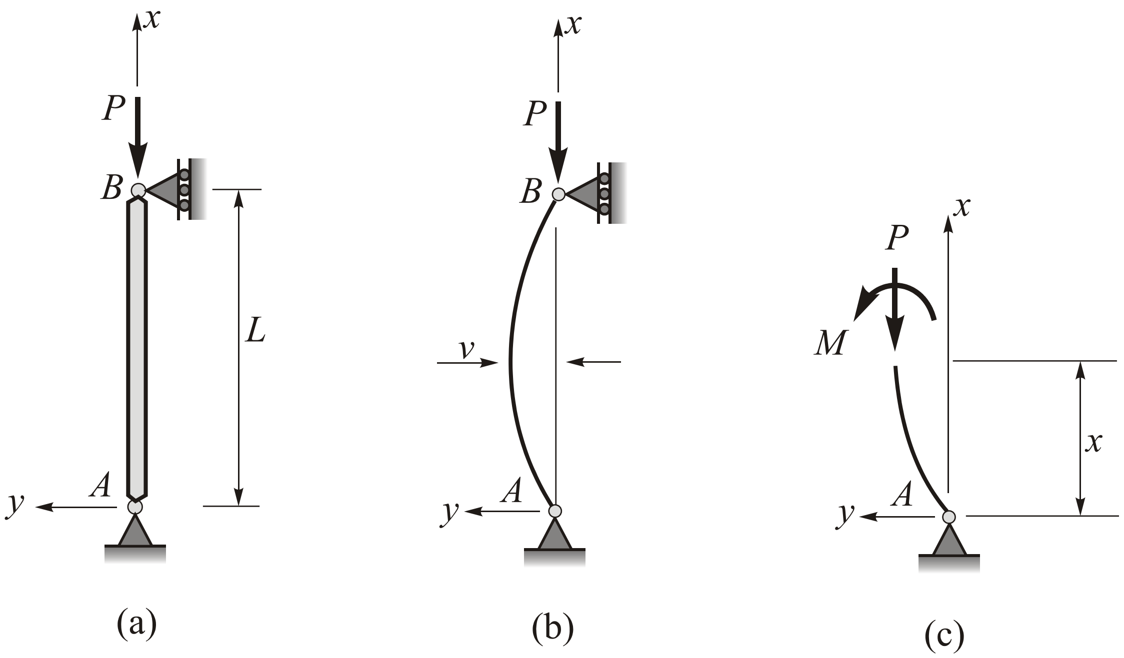


Fig. 5 Column with pinned ends: (a) ideal column, (b) buckled shape, and (c) axial force P and bending moment M acting at a cross section

When the axial load P has a small value, the column remains perfectly straight and undergoes direct axial compression. The only stresses are the uniform compressive stresses obtained from the equation $\sigma = P/A$. Under low loading, the column is in **stable equilibrium**, which means that it returns to the straight position after a disturbance. For instance, if we apply a small lateral load and cause the column to bend, the deflection will disappear and the column will return to its original position when the lateral load is removed.

As the axial load P is gradually increased, we reach a condition of **neutral equilibrium** (similar to Fig. 4) in which the column may have a bent shape. The corresponding value of the load is the **critical load** P_{cr}. At this load the column may undergo small lateral deflections with no change in the axial force. For instance, a small lateral load will produce a bent shape that does not disappear when the lateral load is removed. Thus, the ***critical load can maintain the column in equilibrium either in the straight position or in a slightly bent position***.

At higher values of the load, the column is unstable and may collapse by buckling in result of excessive bending. For the ideal case, the column will be in equilibrium in the straight position even when the axial force P is greater than the critical load. However, since the equilibrium is unstable, the smallest imaginable disturbance will cause the column to deflect sideways. Once that happens, the deflections will immediately increase and the column will fail by buckling.

The behavior of an ideal column compressed by an axial load P (Figs. 5a and b) may be summarized as follows:

If $P < P_{cr}$, the column is in **stable equilibrium** in the straight position.

If $P = P_{cr}$, the column is in **neutral equilibrium** in either the straight or a slightly bent position.

If $P > P_{cr}$, the column is in **unstable equilibrium** in the straight position and will buckle under the slightest disturbance.

1.1 Differential Equation for Column Buckling

To determine the critical loads and corresponding deflected shapes for an ideal pin-ended column (Fig. 5a), we will use the simplest of differential equations of the deflection curve of a beam. These equations are applicable to a buckled column because the column bends similar as a beam (Fig. 5b). We will use the second-order equation (the bending-moment equation) because its general solution is the simplest. It is valid under stress level not exceeding the proportionality limit $\left(\sigma < \sigma_{pr}\right)$:

$$EIv'' = M\,, \tag{3}$$

in which M is the bending moment at any cross section, v is the **lateral deflection** in the direction, and EI is the **flexural rigidity for bending** in the xz plane.

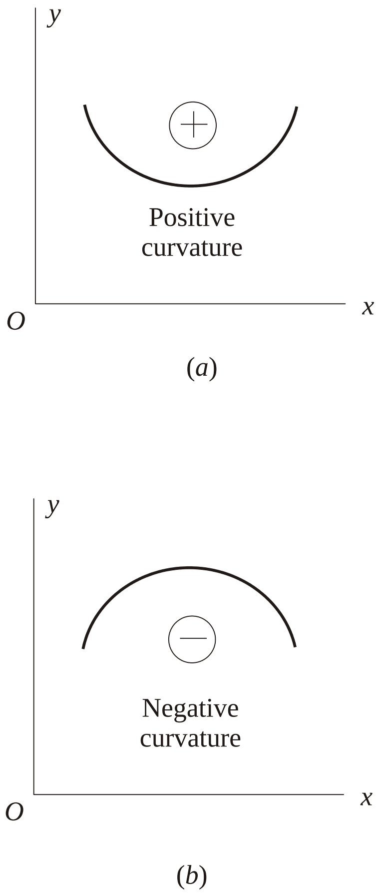


Fig. 6 Sign convention for curvature

The bending moment M at an arbitrary distance x from end A of the buckled column is shown acting in its positive direction in Fig. 4c, because we will believe that ***positive bending moment produces positive curvature*** (see Figs. 6 and 7). The axial force P acting at the cross section is also shown in Fig. 5. Since there are no horizontal forces acting at the supports, there are no shear forces in the column. Therefore, from equilibrium of moments about point A (see Fig. 5c), we obtain

$$M + Pv = 0 \quad \text{or} \quad M = -Pv\,, \tag{4}$$

where v is the deflection at the cross section.

The same expression for the bending moment is obtained if we assume that the column buckles to the right instead of to the left (Fig. 8a). When the column deflects to the right, the deflection itself is $-v$ but the moment of the axial force about point A also changes sign. Thus, the equilibrium equation for moments about point A (see Fig. 8b) is

$$M - P\left(-v\right) = 0\,, \tag{5}$$

which gives the same expression for the bending moment M as before.

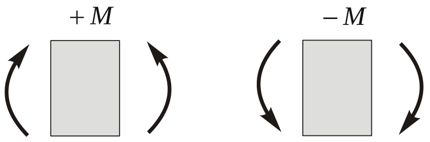


Fig. 7 Sign conventions for bending moment *M* (positive moment produces positive curvature if *z*-axis is directed up)

The differential equation of the deflection curve (Eq. 3) now becomes

$$EIv''+Pv=0, \quad \text{or} \quad EI\frac{d^2v}{dx^2}+Pv=0. \qquad (6)$$

By solving this equation, which is a ***homogeneous, linear, differential equation of second order with constant coefficients***, we can determine the magnitude of the **critical load** and the **deflected shape** of the buckled column.

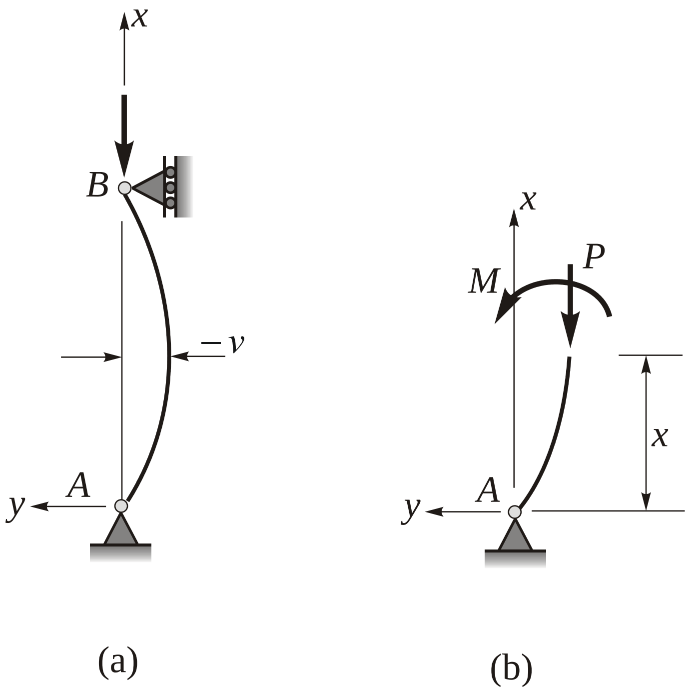


Fig. 8 Column with pinned ends (alternative direction of buckling)

Despite of that we investigate the buckling of columns by solving the same basic differential equation as are used in the calculations of beam deflections, the fundamental difference in these two types of analysis is evident. ***In the case of beam deflections, the bending moment M*** appearing in Eq. (3) ***is a function of the load only*** – it does not depend upon the deflections of the beam (***assumption on relative rigidity of a rod***). In ***buckling, the bending moment is a function of the deflections themselves*** (Eq. 4). In bending theory, the deflected shape of the structure was not considered, and the equations of equilibrium were based upon the geometry of the **undeformed**

structure. Now, however, the geometry of the **deformed structure** is taken into account when writing equations of equilibrium.

1.2 Solution of the Differential Equation

Setting the notation

$$k^2 = \frac{P}{EI} \quad \text{or} \quad k = \sqrt{\frac{P}{EI}}, \tag{7}$$

in which *k* is always taken as a positive quantity (*k* has units of the reciprocal of length), we can rewrite Eq. (6) in the form

$$v''(x) + k^2 v(x) = 0, \quad \text{or} \quad \frac{d^2 v}{dx^2} + k^2 v = 0. \tag{8}$$

The solution of this equation is

$$v(x) = C_1 \sin kx + C_2 \cos kx, \tag{9}$$

which is a form of the familiar equation of simple harmonic motion. The constants of integration C_1 and C_2 are evaluated from the boundary conditions at the ends of the column (note that the number of arbitrary constants in the solution (two in this case) agrees with the order of the differential equation): the deflection is zero when $x = 0$ and $x = L$ (see Fig. 1b):

$$v(0) = 0, \quad v(L) = 0. \tag{10}$$

The first condition gives $C_2 = 0$, and therefore

$$v(x) = C_1 \sin kx. \tag{11}$$

The second condition gives

$$C_1 \sin kL = 0. \tag{12}$$

From this equation we conclude that either $C_1 = 0$ or $\sin kL = 0$. We will consider both of these possibilities.

1. If the constant C_1 equals zero, the deflection v is also zero (see Eq. 11), and therefore the column remains straight. In addition, when C_1 equals zero, Eq. (12) is satisfied for any value of the quantity kL. Consequently, the axial load P may also have any value (see Eq. 7b).

2. The second possibility for satisfying Eq. (12) is given by the following equation, known as the **buckling equation**:

$$\sin kL = 0. \tag{13}$$

This equation is satisfied when $kL = 0,\ \pi,\ 2\pi, \ldots$. However, since $kL = 0$ means that $P = 0$, this solution is not of interest. Therefore, the solutions we will consider are:

$$kL = \pi n \quad (n = 1, 2, 3, \ldots), \tag{14}$$

or (see Eq. 7a):

$$P = \frac{n^2 \pi^2 EI}{L^2} \quad (n = 1, 2, 3, \ldots). \tag{15}$$

This formula gives the values of P that satisfy the buckling equation and provide solutions (other than the trivial solution) to the differential equation.

The equation of the deflection curve (from Eqs. 11 and 14) is

$$v(x) = C_1 \sin kx = C_1 \sin \frac{n\pi x}{L} \quad (n = 1, 2, 3, \ldots). \tag{16}$$

Only when P has one of the values given by Eq. (15) it is theoretically possible for the column to have a bent shape (see Eq. 16). Therefore, the values of P given by Eq. (15) are the critical loads for the column.

1.3 Euler's Formula for Critical Loads

Only the value for $n = 1$ has physical significance, as it determines the smallest value of P for which a buckling shape can occur under static loading. Thus the **critical load** for a column with a pinned end is

$$P_{cr} = \frac{\pi^2 EI}{L^2}, \tag{17}$$

where L represents the **original length of the column**. The preceding result is attributable to Leonhard Euler (1707–1783). He was the first who investigated the buckling of a slender column and determined its critical load. He published his results in 1744.

Eq. (17) is known as **Euler's formula**, the corresponding load is called the **Euler buckling load**. In Eq. (17), EI is the **flexural rigidity for bending** in the *xy* plane, taken to be the plane of buckling owing to the restraints imposed by the end connections (Fig. 1a). However, if the column is free to deflect in any direction, it will tend to bend about the axis having the **smallest principal centroidal moment of inertia** *I*.

The corresponding to P_{cr} **buckled shape** (syn. a **mode shape**) is

$$v(x) = C_1 \sin\frac{\pi x}{L}, \tag{18}$$

as shown in Fig. 7b. ***The constant C_1 represents the deflection at the midpoint of the column*** and may have any small value, either positive or negative.

By taking higher values of the index n in Eqs. (15) and (16), ***we obtain an infinite number of critical loads and corresponding mode shapes***. The mode shape for $n = 2$ has two half-waves, as pictured in Fig. 8c. ***The corresponding critical load is four times larger than the Euler's critical load***. The magnitudes of the critical loads are proportional to the square of *n*, and the number of half-waves in the buckled shape is equal to *n*. Buckled shapes for the higher modes are often of no practical interest because the column buckles when the axial load *P* reaches its lowest critical value. The only way to obtain modes of buckling higher than the first is to provide lateral support of the column at intermediate points, such as at the midpoint of the column shown in Fig. 9.

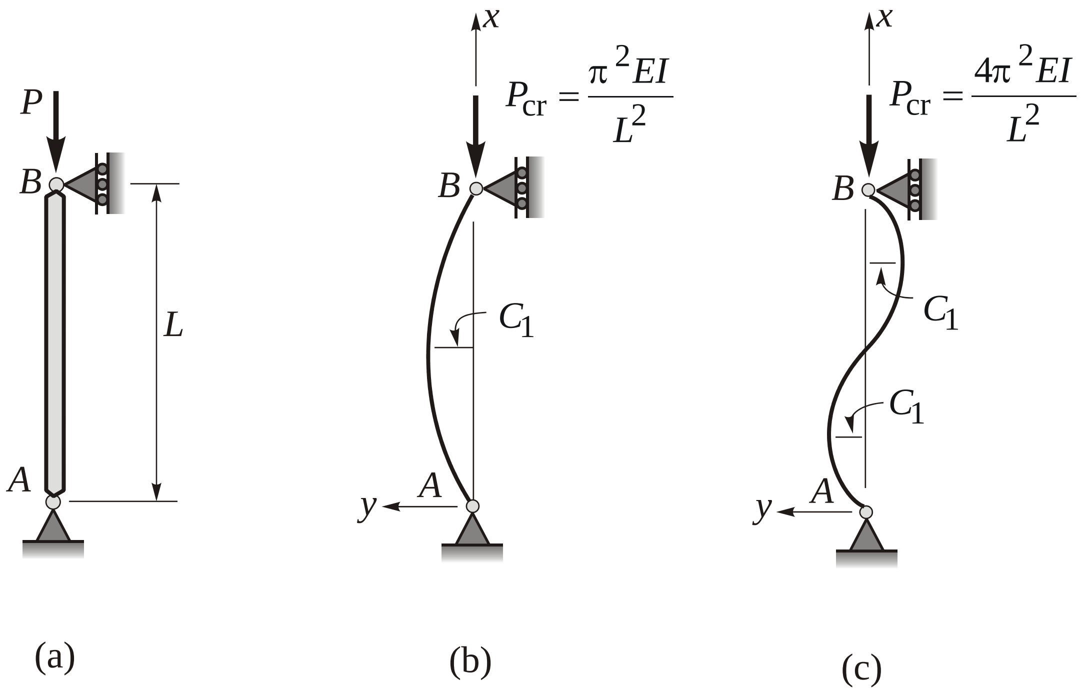


Fig. 9 Buckled shapes for an ideal column with pinned ends: (a) initially straight column, (b) buckled shape for $n = 1$, and (c) buckled shape for $n = 2$

From Eq. (16) we see that the ***critical load of a column is proportional to the flexural rigidity EI and inversely proportional to the square of the length***. It's interesting that the ***strength of the material*** (represented by a quantity such as the proportional limit or the yield stress), ***does not appear in the equation for the critical load***. Therefore, ***increasing a strength property does not raise the critical load of a slender column. It can only be raised by increasing the flexural rigidity, reducing the length, or providing additional lateral support***.

The flexural rigidity can be increased by using a material with larger modulus of elasticity E or by distributing the material in such a way as to increase the moment of inertia I of the cross section, just as a beam can be made stiffer by increasing the moment of inertia. The moment of inertia is increased by distributing the material farther from the neutral axis of the cross section. Hence, a hollow tubular member is generally more economical for use as a column than a solid member having the same cross-sectional area. Reducing the wall thickness of a tubular member and increasing its lateral dimensions (while keeping the cross-sectional area constant) also increases the critical load because the moment of inertia is increased. This process has a practical limit, however, because eventually the wall itself will become unstable. When that happens, ***localized buckling occurs in the form of small wrinkles*** in the walls of the column. Thus, we must distinguish between **overall buckling** of a column, and **local buckling** of its parts.

Earlier we assumed that the *xy* plane was a plane of symmetry of the column and that buckling took place in that plane (Fig. 9). This assumption will be met if the column has lateral supports perpendicular to the plane of the figure, so that the column is constrained to buckle in the *xy* plane. If the column is supported only at its ends and is free to buckle in any direction, then bending will occur about the principal centroidal axis having the smaller moment of inertia. For instance, consider the rectangular, wide-flange, and channel cross sections shown in Fig. 10. In each case, the moment of inertia I_x is greater than the moment of inertia I_y; hence the column will buckle in the *x*-*x* plane, and the smaller moment of inertia I_y should be used in the formula for the critical load. If the cross section is square or circular, all centroidal axes have the same moment of inertia and buckling may occur in any plane.

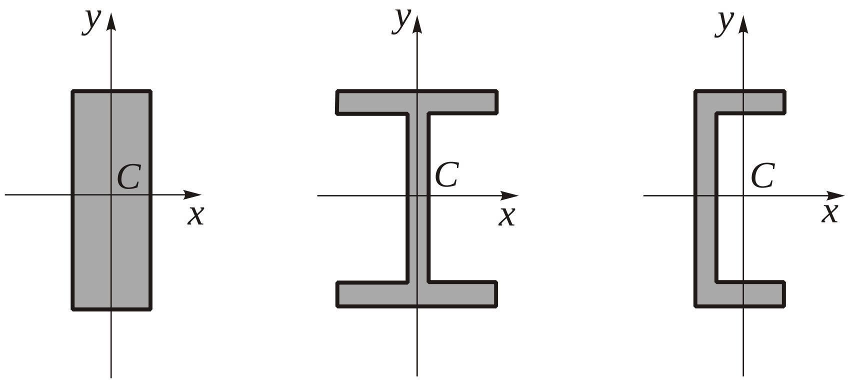


Fig. 10 Cross sections of columns showing principal centroidal axes with $I_x > I_y$

1.4 Critical Stress

If we know the critical load for a column, we can calculate the corresponding **critical stress** by dividing the critical load by the cross-sectional area. For the fundamental case of buckling (Fig. 10b), the critical stress is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2}, \tag{19}$$

in which I is the moment of inertia for the principal axis about which buckling occurs. This equation can be written in a more useful form by introducing the term

$$i = \sqrt{\frac{I}{A}}, \tag{20}$$

in which i is the **radius of gyration** of the cross section in the plane of bending. Then the equation for the critical stress becomes

$$\sigma_{cr} = \frac{\pi^2 E}{(L/i)^2}, \tag{21}$$

in which L/i is a dimensionless ratio called the **slenderness ratio** λ:

$$\lambda = \frac{L}{i}. \tag{22}$$

After this, Euler's formula Eq. (17) becomes

$$P_{cr} = \frac{\pi^2 EA}{(L/i)^2} = \frac{\pi^2 EA}{\lambda^2}. \tag{23}$$

The slenderness ratio depends only on the dimensions of the column, including length and cross section. A column that is long and slender will have a high slenderness ratio and therefore a low critical force and stress. A short column will have a low slenderness ratio and will buckle at a high stress. Typical values of slenderness ratio for actual columns are between 30 and 200.

A graph of critical stress as a function of the slenderness ratio is known as **Euler's curve** (Fig. 11). The curve shown in the figure is plotted for a structural steel with $E = 207$ GPa. The ***curve is valid only when the critical stress is less than the proportional limit*** of the steel, because the equations were derived using Hooke's law. Therefore, we draw a horizontal line on the graph at the proportional limit of the steel (assumed to be 250 MPa) and limit Euler's curve at that level of stress. Really ***Euler's curve is a plot of an equation of the third degree in two variables***.

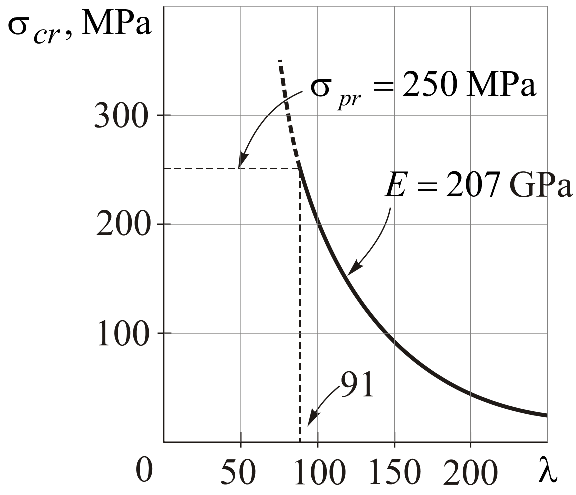


Fig. 11 Graph of Euler's curve (from Eq. 21) for structural steel with $E = 207$ GPa and $\sigma_{pr} = 250$ MPa

1.5 Influence of Large Deflections and Imperfections on Critical Force

The equations for critical loads mentioned above were created for **ideal columns**, that is, columns for which the loads are precisely applied, the construction is perfect, and the material follows Hooke's law. In result we found that the magnitudes of the small deflections at buckling were undefined. Thus, when $P = P_{cr}$, the column may have any small deflection. The ***theory for ideal columns is limited to small deflections*** because we used the second derivative v'' for the curvature. A more exact analysis is based upon the exact expression for curvature. The basic relationship stating that the curvature of a beam is proportional to the bending moment $\left(k = 1/\rho = M/EI\right)$ was first attained by Jacob Bernoulli, although he obtained an incorrect value for the constant of proportionality. The relationship was used later by Leonhard Euler, who solved the differential equation of the deflection curve for both large deflections, using $k = 1/\rho = v''\Big/\left[1+\left(v'\right)^2\right]^{3/2}$ and small deflections using $k = 1/\rho = v''$.

A more exact analysis shows that there is no indefiniteness in the magnitudes of the deflections at buckling. Instead, for an ideal, linearly elastic column, the load-deflection diagram goes upward in accordance with curve *B* of Fig. 12. Thus, after a linearly elastic column begins to buckle, ***an increasing load is required to cause an increase in the deflections***.

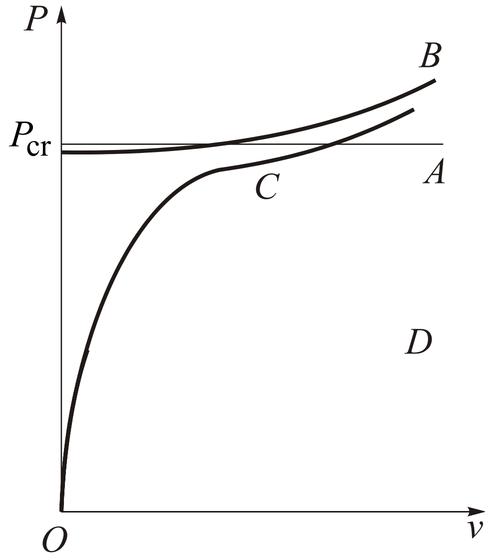


Fig. 12 Load-deflection diagram for columns: line *A*, ideal elastic column with small deflections; curve *B*, ideal elastic column with large deflections; curve *C*, elastic column with imperfections

Next problem is connected with the column **non-perfection**. If for instance, the column has an **imperfection** in the form of a small initial curvature, i.e. the unloaded column is not perfectly straight. Such imperfections produce deflections from the onset of loading, as shown by curve *C* in Fig. 12. For small deflections, curve *C* approaches line *A* as an asymptote. However, as the deflections become large, it approaches curve *B*. The larger the imperfections, the further curve *C* moves to the right, away from the vertical line. Conversely, if the column is constructed with considerable accuracy, curve *C* approaches the vertical axis and the horizontal line labeled *A*. By comparing lines *A*, *B*, and *C*, we see that for practical purposes the critical load represents the maximum load-carrying capacity of an elastic column, because large deflections are not acceptable in most applications.

1.6 Optimum Shapes of Columns

First of all, we will we analyze only prismatic columns, because compressed members usually have the same cross sections throughout their lengths. Prismatic columns are not the optimum shape if minimum weight is desired. The critical load of a column consisting of a given amount of material may be increased by varying the shape so that the ***column has larger cross sections in those regions where the bending moments are larger***. Consider, for instance, a column of solid circular cross section with pinned ends. A column shaped as shown in Fig. 13a will have a larger critical load than a prismatic column made from the same volume of material. As a means of approximating this optimum shape, prismatic columns may be reinforced over part of their lengths (Fig. 13b).

Now consider a prismatic column with pinned ends that is free to buckle in any lateral direction. Let assume that the column has a solid cross section, such as a circle, square, triangle, rectangle, or hexagon (Fig. 14). Problem is to find the cross section which gives the largest critical load. To solve it we will calculate the critical load from the Euler formula $P_{cr} = \pi^2 EI / L^2$ using the smallest moment of inertia for the cross section, which will be different for all cross sections under consideration.

The circle shape as answer is common, but not correct. The solutions show that a cross section in the shape of an equilateral triangle gives a 21% higher critical load than does a circular cross section of the same area. The critical load for an equilateral triangle is also higher than the loads obtained for the other shapes; hence, an ***equilateral triangle is the optimum cross section*** (based only upon theoretical considerations).

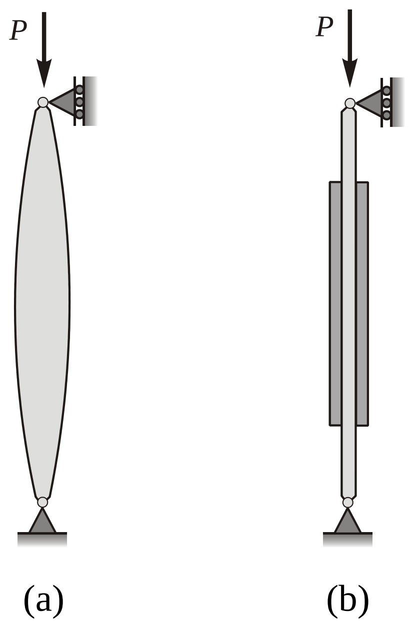


Fig. 13 Nonprismatic columns

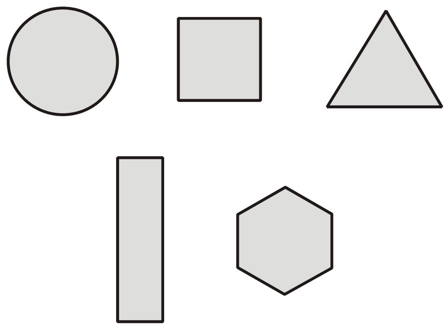


Fig. 14 Different cross-sectional shapes for a prismatic column to select the most stable at specified weight

2 Classification of Columns and Limitations on Euler's Formula

The behavior of an ideal column was earlier represented as a plot of average compressive stress P/A versus slenderness ratio $\lambda = L/i$ (Fig. 15), which is based on Euler’s curve (Eq. 21, Fig. 11). Such a representation offers a clear basis for the classification of compressed bars. Tests of columns verify each portion of the curve. The range of λ is a function of the material under consideration.

2.1 Long Columns

For a ***long column***, that is, ***a member of sufficiently large slenderness ratio, buckling occurs elastically at a stress that does not exceed the proportional limit of the material***. Hence the Euler's load is appropriate to this case, and the **critical stress** is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/i)^2}. \tag{24}$$

The corresponding portion CD of the curve (Fig. 15) was earlier labeled as **Euler's curve**. The critical value of slenderness ratio that fixes the lower limit of this curve is found by equating σ_{cr} to the proportional limit (σ_{pr}) of the specified material:

$$\left(\frac{L}{i}\right)_{cr} = \sqrt{\frac{\pi^2 E}{\sigma_{pr}}} = \lambda_{cr}. \tag{25}$$

For example, in the case of a **structural steel** with $E = 207\,\text{GPa}$ and $\sigma_{pr} = 250\,\text{MPa}$ (see Fig. 10), the foregoing results in $(L/i)_{cr} = 91$. The term in (Eq. 25) is called **critical slenderness ratio** λ_{cr}. ***Above this value, an ideal column buckles elastically and the Euler load is valid***. ***Below this value, the stress in the column exceeds the proportional limit and the column buckles inelastically***.

From Fig. 15 we observe that very slender columns buckle at low levels of critical stress; they are much less stable than short columns.

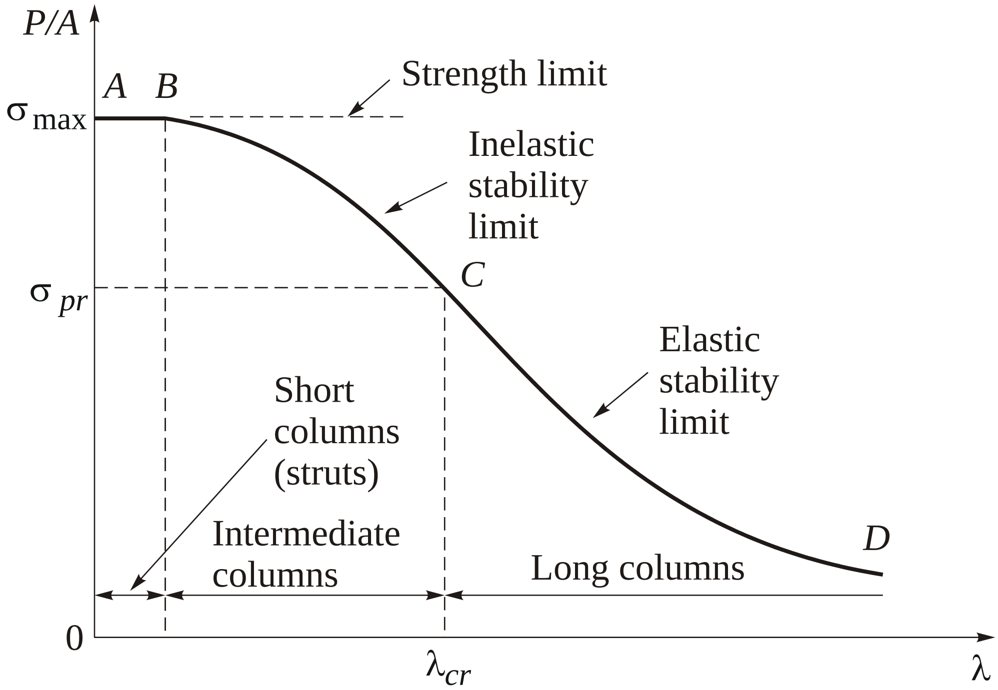


Fig. 15 Average stress in columns versus slenderness ratio

2.2 Short Columns

Compressed members having low slenderness ratios (for example, steel rods with $L/i<30$) exhibit essentially no instability and are referred to as **short columns** or **struts**. ***For these bars, failure occurs by yielding or by crushing, without buckling, at stresses exceeding the proportional limit of the material***. Thus the maximum stress

$$\sigma_{\max}=\frac{P}{A} \tag{27}$$

is the **strength limit** (**failure stress**) of such a column, represented by horizontal line *AB* in Fig. 15. It is equal to the **yield stress** σ_y for ductile materials or **ultimate stress in compression** σ_u for brittle materials.

2.3 Intermediate Columns

Most structural columns lie in a region between short and long classifications. ***Such intermediate-length columns do not fail by direct compression or by elastic instability***. Consequently, Eqs. (25) and (27) do not apply, and a additional ***non-elastic analysis is required***. ***The failure of an intermediate column occurs by inelastic***

buckling at stress levels exceeding the proportional limit. The particularities of inelastic buckling will be considered in future.

3 Columns with other Support Conditions and Concept of an Effective Length

It is evident that the critical load is dependent upon the end restraints. Buckling of a column with pinned ends is usually considered as the most basic case of buckling. However, in practice we deal with many other end conditions, such as **fixed ends**, **free ends**, and **elastic supports**. ***The critical loads for columns with various kinds of support conditions can be determined from the differential equation of the deflection curve***.

The procedure is as follows. First, with the column assumed to be in the buckled state, we obtain an expression for the bending moment in the column. Second, we set up the differential equation of the deflection curve, using the bending-moment equation $(EIv'' = M)$. Third, we solve the equation and obtain its general solution, which contains two constants of integration plus other unknown quantities. Fourth, we apply boundary conditions pertaining to the deflection v and the slope v' and obtain a set of simultaneous equations. Finally, we solve those equations to obtain the critical load and the deflected shape of the buckled column. This procedure is illustrated for three different types of columns.

3.1 Column Fixed at the Base and Free at the Top

We will consider an ideal column that is fixed at the base, free at the top, and subjected to an axial load P (Fig. 16a). This column is one first analyzed by L. Euler in 1744. The deflected shape of the buckled column is shown in Fig. 16b. From this figure we see that the bending moment at distance x from the base is

$$M = P(\delta - v), \tag{28}$$

where δ is the deflection at the free end of the column. The differential equation of the deflection curve then becomes

$$EIv'' = M = P(\delta - v), \tag{29}$$

in which I is the moment of inertia for buckling in the xy plane.

Using the notation $k^2 = P/EI$ (Eq. 7), we can rewrite Eq. (29) into the form

$$v'' + k^2 v = k^2 \delta, \tag{30}$$

which is a ***linear differential equation of second order with constant coefficients***. It is a more complicated equation than the equation for a column with pinned ends (see Eq. 8) because it has a nonzero term on the right-hand side. The general solution of Eq. (29) consists of two parts: (1) the **homogeneous solution**, which is the solution of the homogeneous equation obtained by replacing the right-hand side with zero, and (2) the **particular solution**, which is the solution of Eq. (29) that produces the term on the right-hand side.

The homogeneous solution (also called the **complementary solution**) is the same as the solution of Eq. (8); hence

$$v_h(x) = C_1 \sin kx + C_2 \cos kx, \tag{31}$$

where C_1 and C_2 are constants of integration. Note that when v_H is substituted into the left-hand side of the differential equation (Eq. 29), it produces zero.

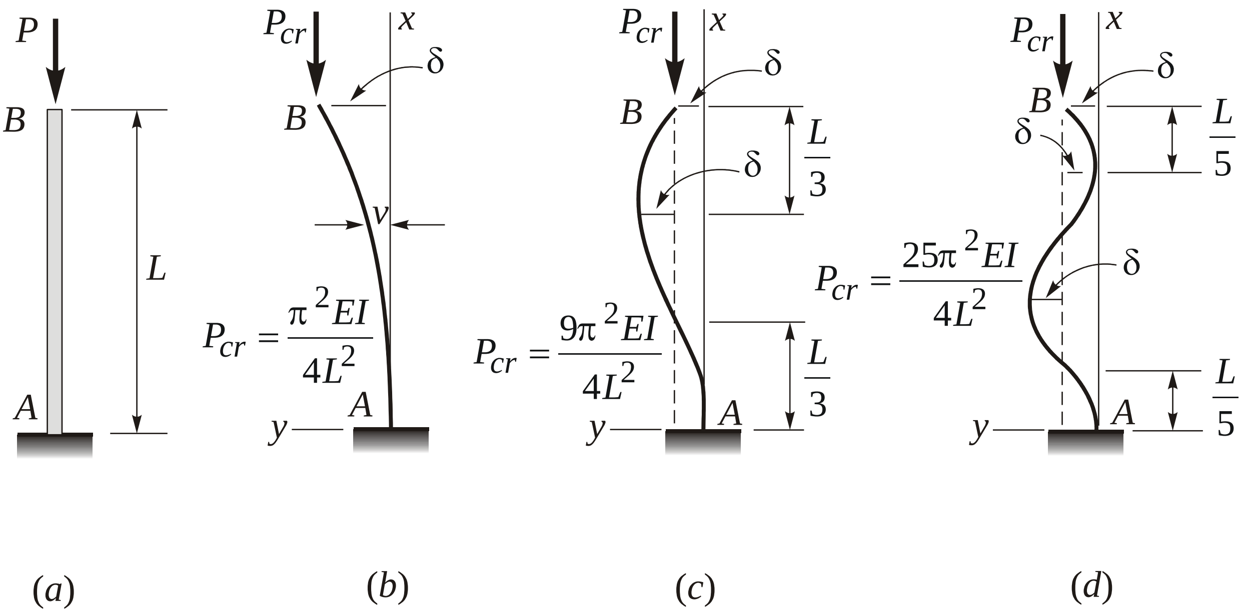


Fig. 16 Ideal column fixed at the base and free at the top: (a) initially straight column, (b) buckled shape for $n = 1$, (c) buckled shape for $n = 3$, and (d) buckled shape for $n = 5$

The particular solution of the differential equation is

$$v_p = \delta . \tag{32}$$

When v_p is substituted into the left-hand side of the differential equation, it produces the right-hand side, that is, it produces the term $k^2\delta$. Consequently, the general solution of the equation, equal to the sum of v_h and v_p, is

$$v(x) = C_1 \sin kx + C_2 \cos kx + \delta . \tag{33}$$

This equation contains three unknown quantities (C_1, C_2 and δ), and therefore three boundary conditions are needed to complete the solution.

At the base of the column, the deflection and slope are each equal to zero. Therefore, we obtain the following boundary conditions:

$$v(0) = 0 , \quad v'(0) = 0 . \tag{34}$$

Applying the first condition to Eq. (32), we find

$$C_2 = -\delta . \tag{4.8}$$

To apply the second condition, we first differentiate Eq. (32) to obtain the slope:

$$v'(x) = C_1 k \cos kx - C_2 k \sin kx . \tag{35}$$

Applying the second condition to this equation, we find $C_1 = 0$.

Now we can substitute the expressions for C_1 and C_2 into the general solution (Eq. 32) and obtain the equation of the deflection curve for the buckled column:

$$v(x) = \delta(1 - \cos ks) . \tag{36}$$

Note that ***this equation gives only the shape of the deflection curve*** – the amplitude δ remains undefined. Thus, when the column buckles, the deflection given by Eq. (36) may have any arbitrary magnitude, except that it must remain small (because the differential equation is based upon small deflections).

The third boundary condition applies to the upper end of the column, where the deflection v is equal to δ:

$$v(L) = \delta . \quad (37)$$

Using this condition with Eq. (36), we get

$$\delta \cos kL = 0 . \quad (38)$$

From this equation we conclude that either $\delta = 0$ or $\cos kL = 0$. If $\delta = 0$, there is no deflection of the bar (see Eq. 36) and we have the trivial solution – the column remains straight and buckling does not occur. In that case, Eq. (38) will be satisfied for any value of the quantity *kL*, that is, for any value of the load *P*.

The other possibility for solving Eq. (38) is

$$\cos kL = 0 , \quad (39)$$

in which case Eq. (38) is satisfied regardless of the value of the deflection δ. Thus, as already observed, δ is undefined and may have any small value. The equation $\cos kL = 0$, which is the **buckling equation**, is satisfied when

$$kL = \frac{n\pi}{2} \quad (n = 1, 3, 5, ...) . \quad (40)$$

Using the expression $k^2 = P/EI$, we obtain the following formula for the critical loads:

$$P_{cr} = \frac{n^2 \pi^2 EI}{4L^2} \quad (n = 1, 3, 5, ...) \quad (41)$$

Also, the buckled mode shapes are obtained from Eq. (36):

$$v = \delta \left(1 - \cos \frac{n\pi x}{2L} \right) \quad (n = 1, 3, 5, ...) . \quad (42)$$

The lowest critical load is obtained by substituting $n = 1$ in Eq. (41):

$$P_{cr} = \frac{\pi^2 EI}{4L^2} . \quad (43)$$

The corresponding buckled shape (from Eq. 42) is

$$v(x) = \delta\left(1 - \cos\frac{\pi x}{2L}\right), \tag{44}$$

it is shown in Fig. 16b.

By taking higher values of the index n, we can theoretically obtain an infinite number of critical loads from Eq. (41). The corresponding buckled mode shapes have additional waves in them. For instance, when $n = 3$ the buckled column has the shape shown in Fig. 16c and P_{cr} is nine times larger than for $n = 1$. Similarly, the buckled shape for $n = 5$ has even more waves (Fig. 16d) and the critical load is twenty-five times larger.

3.2 Effective Lengths of Columns

The critical loads for columns with various support conditions can be related to the critical load of a pinned-end column through the concept of an **effective length**. To demonstrate this idea, consider the deflected shape of a column fixed at the base and free at the top (Fig. 17a). This column buckles in a curve that is one-quarter of a complete sine (or cosine) wave. If we extend the deflection curve (Fig. 17b), it becomes one-half of a complete sine wave, which is the deflection curve for a pinned-end column.

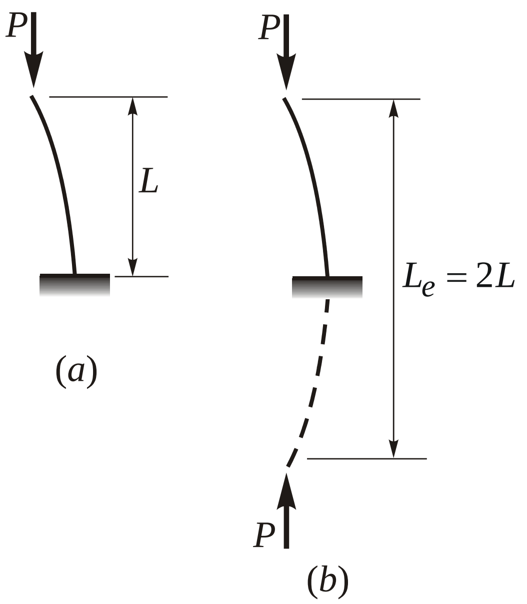


Fig. 17 Deflection curves showing the effective length L_e for a column fixed at the base and free at the top

The ***effective length*** L_e for any column ***is the length of the equivalent pinned-end column***, that is, it is the length of a pinned-end column having a deflection curve that exactly matches all or part of the deflection curve of the original column. Another way of expressing this idea is to say that the effective length of a column is the distance between the inflection points on its elastically deflected curve, or points of zero moment, assuming that the curve is extended (if necessary) until points of inflection are reached. Thus, for a fixed-free column (Fig. 17), the effective length is

$$L_e = 2L. \tag{45}$$

Since the effective length is the length of an equivalent pinned-end column, we can write a general formula for critical loads as follows:

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}. \tag{46}$$

Thus, if we know the effective length of a column (no matter how complex the end conditions may be), we can substitute into this equation and determine the critical load. For instance, in the case of a fixed-free column, we can substitute $L_e = 2L$ and obtain Eq. (43).

The effective length is often expressed in terms of an **effective-length factor** μ:

$$L_e = \mu L, \tag{47}$$

where L is the **actual length of the column**. Thus, the critical load is

$$P_{cr} = \frac{\pi^2 EI}{(\mu L)^2}. \tag{48}$$

The factor μ equals 2 for a column fixed at the base and free at the top and equals 1 for a pinned-end column.

3.3 Column with Both Ends Fixed against Rotation

Let us consider a column with both ends fixed against rotation (Fig. 18a). Upper rigid block in Fig. 18 shows the constraint in such a manner that rotation and horizontal displacement are prevented but vertical movement can occur.

The buckled shape of the column in the first mode is shown in Fig. 18c. The deflection curve is symmetrical (with zero slope at the midpoint) and has zero slope at the ends. Because rotation at the ends is prevented, reactive moments M_0 appear at the supports. These moments, as well as the axial forces, are shown in the figure.

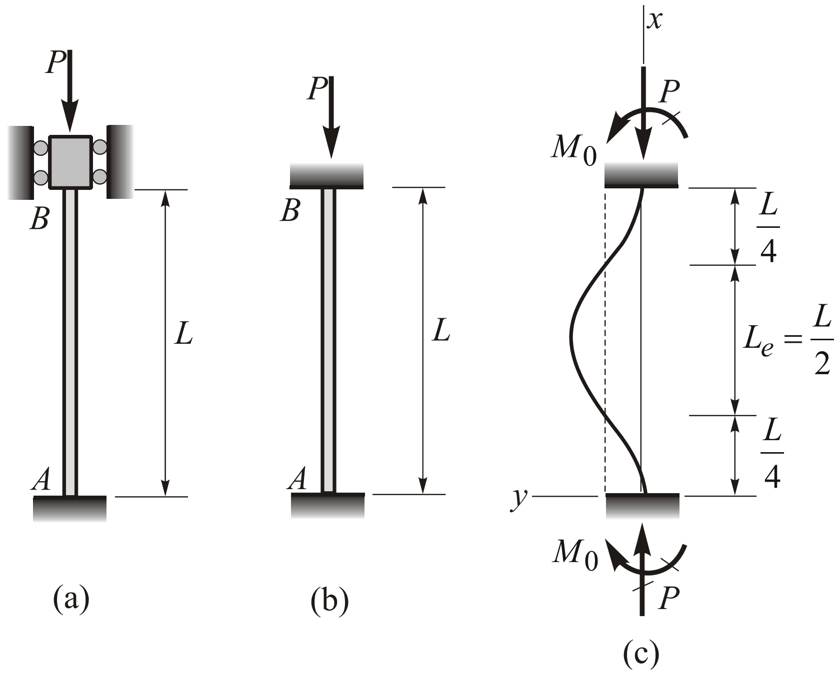


Fig. 18 Buckling of a column with both ends fixed against rotation

From previous solutions of the differential equation, it is evident that the equation of the deflection curve involves sine and cosine functions. Also the curve is symmetric about the midpoint. Therefore, we see that the curve must have inflection points at distances $L/4$ from the ends. It follows that the middle portion of the deflection curve has the same shape as the deflection curve for a pinned-end column. Thus, the effective length of a column with fixed ends, equal to the distance between inflection points, is

$$L_e = \frac{L}{2}. \tag{49}$$

Substituting into Eq. (46) gives the critical load:

$$P_{cr} = \frac{4\pi^2 EI}{L^2}. \tag{50}$$

This formula shows that the critical load for a column with fixed ends is four times that for a column with pinned ends. As a check, this result may be verified by solving the differential equation of the deflection curve.

3.4 Column Fixed at the Base and Pinned at the Top

The critical load and buckled mode shape for a column that is fixed at the base and pinned at the top (Fig. 19a) can be determined by solving the differential equation of the deflection curve. When the column buckles (Fig. 19b), a reactive moment M_0 develops at the base because there can be no rotation at that point. Then, from the equilibrium of the entire column, we know that there must be horizontal reactions R at each end such that

$$M_0 = RL . \tag{51}$$

The bending moment in the buckled column, at distance x from the base, is

$$M = M_0 - Pv - Rx = -Pv + R(L - x) \tag{52}$$

and therefore the differential equation is

$$EIv'' = M = -Pv + R(L - x) . \tag{53}$$

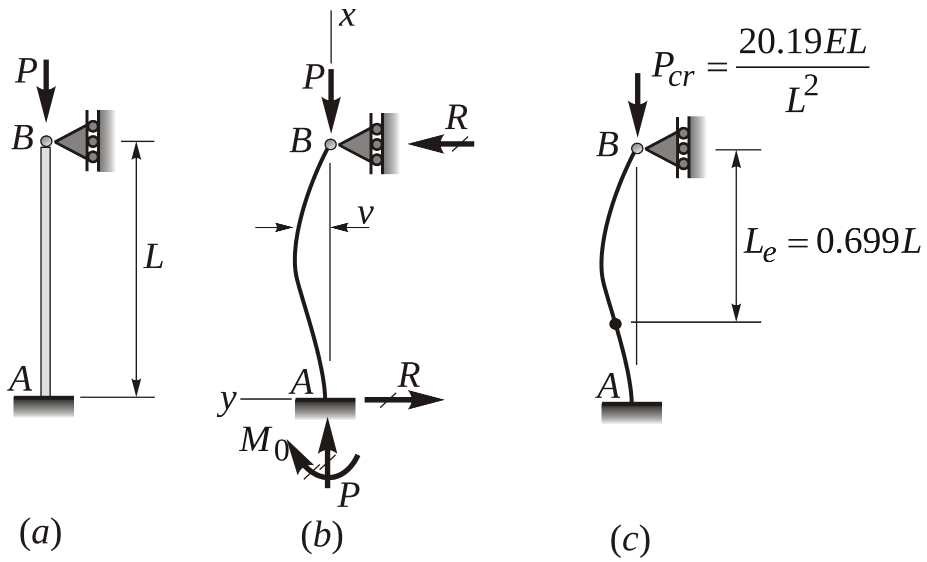


Fig. 19 Column fixed at the base and pinned at the top

Substituting $k^2 = P/EI$ and rearranging, we get

$$v'' + k^2 v = \frac{R}{EI}(L - x) . \tag{54}$$

The general solution of this equation is

$$v(x) = C_1 \sin kx + C_2 \cos kx + \frac{R}{P}(L - x) , \tag{55}$$

in which the first two terms on the right-hand side constitute the homogeneous solution and the last term is the particular solution. As earlier, the general solution can be verified by substitution into the differential equation.

Because the solution contains three unknown quantities (C_1, C_2, and R), we need three boundary conditions. They are

$$v(0)=0, \quad v'(0)=0, \quad v(L)=0. \tag{56}$$

Applying these conditions to Eq. (4.29) yields

$$C_2+\frac{RL}{P}=0, \quad C_1k-\frac{R}{P}=0, \quad C_1\tan kL+C_2=0. \tag{57}$$

All three equations are satisfied if $C_1=C_2=R=0$, in which case we have the trivial solution and the deflection is zero. To obtain the solution for buckling, we must solve Eqs. (57) in a more general manner. One method of solution is to eliminate R from the first two equations, which yields

$$C_1kL+C_2=0 \quad \text{or} \quad C_2=-C_1kL. \tag{58}$$

Next, we substitute this expression for C_2 into last Eq. (57) and obtain the **buckling equation**:

$$kL=\tan kL. \tag{59}$$

The solution of this equation gives the critical load.

Since the buckling equation is a transcendental equation, it cannot be solved plainly. Nevertheless, the values of kL that satisfy the equation can be determined numerically. The smallest nonzero value of kL that satisfies Eq. (59) is

$$kL=4.4934. \tag{60}$$

The corresponding critical load is

$$P_{cr}=\frac{20.19EI}{L^2}=\frac{2.046\pi^2EI}{L^2}, \tag{61}$$

which is higher than the critical load for a column with pinned ends and lower than the critical load for a column with fixed ends (see Eqs. 17 and 50).

The effective length of the column may be obtained by comparing Eqs. (61) and (46); thus,

$$L_e=0.699L\approx 0.7L. \tag{62}$$

This length is the distance from the pinned end of the column to the point of inflection in the buckled shape (Fig. 19c).

The equation of the buckled mode shape is obtained by substituting $C_2 = -C_1 kL$ (Eq. 58) and $R/P = C_1 k$ (Eq. 57) into the general solution (Eq. 55):

$$v(x) = C_1\left[\sin kx - kL\cos kx + k(L - x)\right], \qquad (63)$$

in which $k = 4.4934/L$. The term in brackets gives the mode shape for the deflection of the buckled column. However, the amplitude of the deflection curve is undefined because C_1 may have any value (within the usual limitation that the deflections must remain small).

In addition to the requirement of small deflections, the Euler buckling theory is valid only if the column is perfectly straight and the material follows Hooke's law.

The lowest critical loads and corresponding effective lengths for the four widely used columns are summarized in Fig. 20.

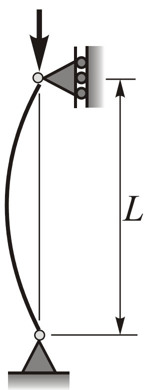
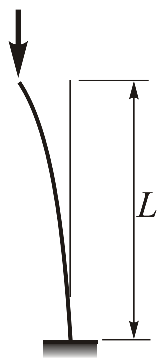
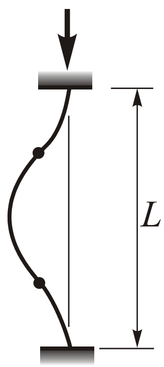
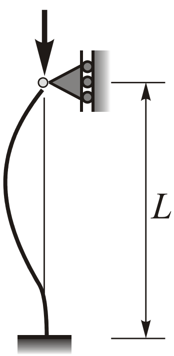
(a) Pinned - pinned column	(b) Fixed - free column	(c) Fixed - fixed column	(d) Fixed - pinned column
$P_{cr} = \frac{\pi^2 EI}{L^2}$	$P_{cr} = \frac{\pi^2 EI}{4L^2}$	$P_{cr} = \frac{4\pi^2 EI}{L^2}$	$P_{cr} = \frac{2.046\pi^2 EI}{L^2}$
			
$L_e = L$	$L_e = 2L$	$L_e = 0.5L$	$L_e = 0.699L$
K=1	K=2	K-0.5	K=0.699

Fig. 20 Critical loads, effective lengths, and effective-length factors for ideal columns