LECTURE 26 Buckling of Columns (Part 2). Inelastic Column Behavior

Introduction

Earlier we analyzed the behavior of columns when the material is stressed ***below the proportional limit***. We began by considering an ideal column subjected to a centrally applied load (Euler buckling), and we arrived at the concept of a critical load P_{cr}. Unfortunately, ***only extremely slender columns remain elastic up to the critical load. More short columns behave inelastically***.

Thus, the maximum load that can be supported by inelastic column may be considerable less that the Euler's load for that some column.

Inelastic buckling is one of the practically important problems of mechanics of materials, that is, the ***buckling of columns when the proportional limit is exceeded***. We will investigate the behavior of the same type of diagram as before, namely, a diagram of average compressive stress P/A versus slenderness ratio L/i (Fig. 1). Note that Euler's curve is shown on this diagram as curve *ECD*. ***This curve is valid only in the region CD where the stress is below the proportional limit*** σ_{pr} of the material. Therefore, the part of the curve above the proportional limit is shown by a dashed line.

The value of slenderness ratio above which Euler's curve is valid (it was called **critical slenderness ratio** λ_{cr}) was obtained as

$$\lambda_{cr} = \left(\frac{L}{i}\right)_{cr} = \sqrt{\frac{\pi^2 E}{\sigma_{pr}}} . \tag{64}$$

For structural steel with $\sigma_{pr} = 250$ MPa and $E = 207$ GPa the critical slenderness ratio is equal to 91.

Above this value, an ideal column buckles elastically and the Euler load is valid. Below this value, the stress in the column exceeds the proportional limit and the column buckles inelastically.

As previously discussed, from Euler's curve we see that long columns with large slenderness ratios buckle at low values of the average compressive stress P/A. ***This condition cannot be improved by using a higher-strength material, because collapse results from instability of the column as a whole and not from failure of the material itself.*** The stress can only be raised by reducing the slenderness ratio L/i or by using a material with higher modulus of elasticity E.

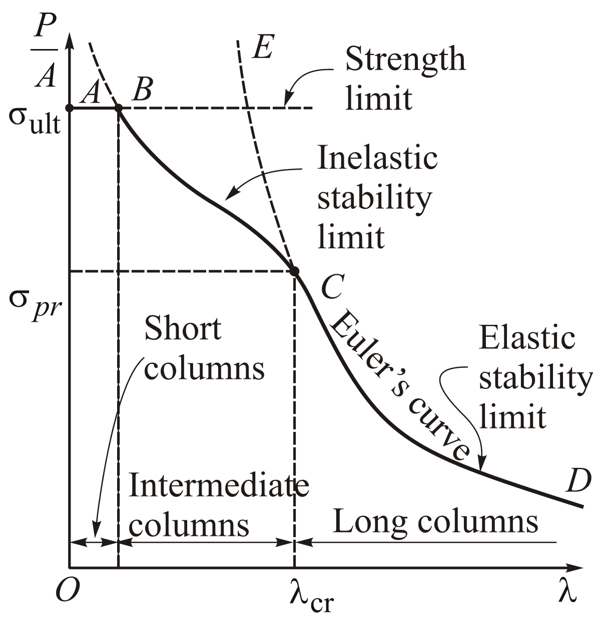


Fig. 1 Diagram of average compressive stress P/A versus slenderness ratio $\lambda = L/i$. *In Euler's curve*, the stress P/A is the actual maximum stress in buckling, i.e. critical stress which decreases if λ increases

When a compressed member is very short, it fails by yielding and crushing of the material, and no buckling or stability considerations are involved. In such a case, we can define an **ultimate compressive stress** σ_{ult} (for brittle materials) or **yield stress** σ_y (for ductile materials) as the failure stress for the material. This stress establishes a strength limit for the column. Evidently, the strength limit is higher than the proportional limit.

Between the regions of short and long columns, there is a range of **intermediate slenderness ratios** too small for elastic stability to govern and too large for strength considerations alone to govern. Such an ***intermediate-length column fails by inelastic buckling, which means that the maximum stresses are above the proportional limit when buckling occurs***. Because the proportional limit is exceeded, the slope of the stress-strain curve for the material is less than the modulus of elasticity; hence the ***critical load for inelastic buckling is always less than the Euler load***.

Fig. 1 shows, that the maximum load-carrying capacity of columns in each of three categories is based upon quite different types of behavior. The maximum load-carrying capacity of a particular column (as a function of its length) is represented by curve *ABCD* in Fig. 1. If the length is very small (region *AB*), the column fails by direct compression; if the column is longer (region *BC*) it fails by inelastic buckling; and if it is even longer (region *CD*), it fails by elastic buckling (that is, Euler buckling). Curve *ABCD* applies to columns with various support conditions if the length L in the slenderness ratio is replaced by the effective length L_e.

The results of load tests on columns are in good agreement with curve *ABCD*. When test results are plotted on the diagram, they generally form a band that lies just below this curve. Unfortunately, considerable scattering of test results is to be expected. To solve the problem of buckling, it is necessary to obtain the **allowable stress** for a column by dividing the maximum stress (from curve *ABCD*) by a given **factor of safety**, which has a value of about 2. It is variable (increasing as L/i increases). Some typical formulas for allowable stresses which allow to design the columns will be given below.

If a column is of intermediate length, the stress in the column will reach the proportional limit before buckling begins (curve *BC* in Fig. 1). ***To calculate critical loads in this intermediate range, a theory of inelastic buckling will be proposed***. Three such theories will be discussed: the **tangent-modulus theory**, the **reduced-modulus theory**, and the **Shanley theory**. These theories illustrate the general steps in development of science on buckling prevention.

1 Historical Note

Leonard Euler was the first who calculated the buckling load (in 1744). The final development of the theory was achieved by Shanley (in 1946). Several famous investigators in the field of mechanics contributed to this development. In 1845 the French engineer A.H.E. Lamarle pointed out that Euler's formula should be used only for slenderness ratios beyond a certain limit and that experimental data should be relied upon for columns with smaller ratios. Then, in 1889, another French engineer, A.G. Considere, published the results of the first tests on columns. He found that the stresses on the concave side of the column increased with so called **tangent modulus** $E_t = d\sigma/d\varepsilon$, and the stresses on the convex side decreased with Young's modulus *E*. Thus, he showed why the Euler formula was not applicable to inelastic buckling, and he stated that the effective modulus was between *E* and E_t. Although he made no attempt to evaluate the effective modulus, Considere was responsible for beginning the **reduced-modulus theory**.

In 1889, the German engineer F. Engesser suggested the **tangent-modulus theory**. He proposed, that E_t be substituted for *E* in Euler's formula for the critical load. Later, in 1895, F. Engesser again presented the tangent-modulus theory, obviously without knowledge of Considere's work. Today, ***the tangent modulus theory is known as the*** **Engesser theory**.

Three months later, the Russian engineer F.S. Jasinsky pointed out that Engesser's tangent-modulus theory was incorrect, called attention to Considere's work, and presented the **reduced-modulus theory**. He also stated that the reduced modulus could not be calculated theoretically. In response, F. Engesser acknowledged the error in the tangent-modulus approach and showed how to calculate the reduced modulus for any cross section. Thus, the reduced-modulus theory is also known as the **Considere-Engesser theory**.

F.S. Jasinsky proposed to connect the points *B* and *C* (Fig. 1) by straight line with the coefficients, calculated in result of experiments. The Jasinsky formula for critical force is the most simple:

$$P_{cr} = A(a - b\lambda), \qquad (65)$$

where *a*, *b* are the coefficients (see Table 1). Evidently, F. Jasinsky formula is correct only for the λ ranged between λ_0 and λ_{cr}.

Table 1

Material	$\sigma_y\left(\sigma_{0,2}\right)$	σ_{pr}	a	b	λ_{cr}	λ_0
	MPa					
Low carbon steel	235	195	310	1.14	100	61
High carbon steel	353	300	440	1.64	85	52
High strength stainless steel	890	750	1100	6.65	58	30
Aluminum alloy	314	250	398	2.78	53	30
Wood (pine)	–	–	28.7	0.19	70	–

The reduced-modulus theory was also presented by the famous scientist T. von Karman in 1908 and 1910, apparently independently of the earlier investigations. He derived the formulas for reduced modulus E_r for both rectangular and idealized wide-flange sections (that is, wide-flange sections without a web).

The reduced-modulus theory was the accepted theory of inelastic buckling until 1946, when the F.R. Shanley pointed out the logical paradoxes in both the tangent-modulus and reduced-modulus theories. Shanley not only explained what was wrong with the generally accepted theories but also proposed his own theory that resolved the paradoxes. He also gave further analyses to support his earlier theory and gave results from tests on columns. Since that time, many other investigators have confirmed and expanded Shanley's concept.

2 Tangent-Modulus Theory

Let us consider an ideal, pinned-end column subjected to an axial force P (Fig. 2). The column is assumed to have a slenderness ratio $\lambda = L/i$ that is less than the critical slenderness ratio λ_{cr}, and therefore the ***axial stress P/a reaches the proportional limit before the critical load is reached***.

The compressive stress-strain diagram for the material of the column is shown in Fig. 3. The proportional limit of the material is indicated as σ_{pr}, and the actual stress σ_A in the column (equal to P/A) is represented by point *A* (which is above the proportional limit). If the load is increased, so that a small increase in stress occurs, the relationship between the increment of stress and the corresponding increment of strain is given by the slope of the stress-strain diagram at point *A*. This slope, equal to the slope of the tangent line at *A*, is called the **tangent modulus** and is denoted by E_t; thus,

$$E_t = \frac{d\sigma}{d\varepsilon}. \qquad (66)$$

It is important to note that the ***tangent modulus decreases as the stress increases beyond the proportional limit. When the stress is below the proportional limit, the tangent modulus is the same as the ordinary elastic modulus E***.

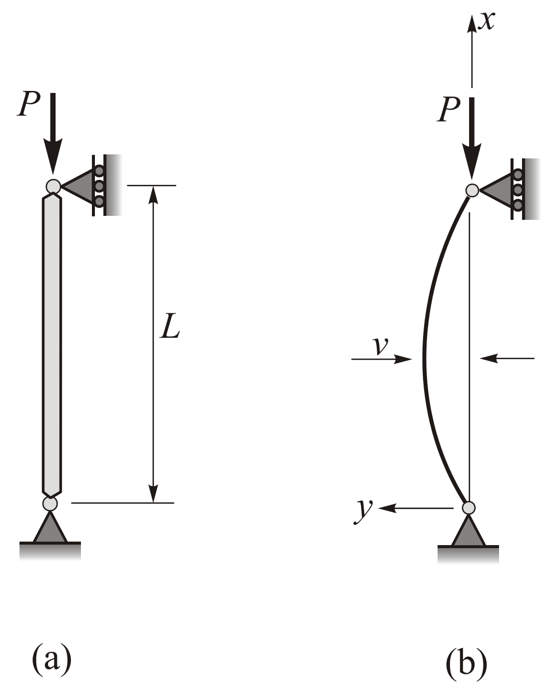


Fig. 2 Ideal column of intermediate length that buckles inelastically

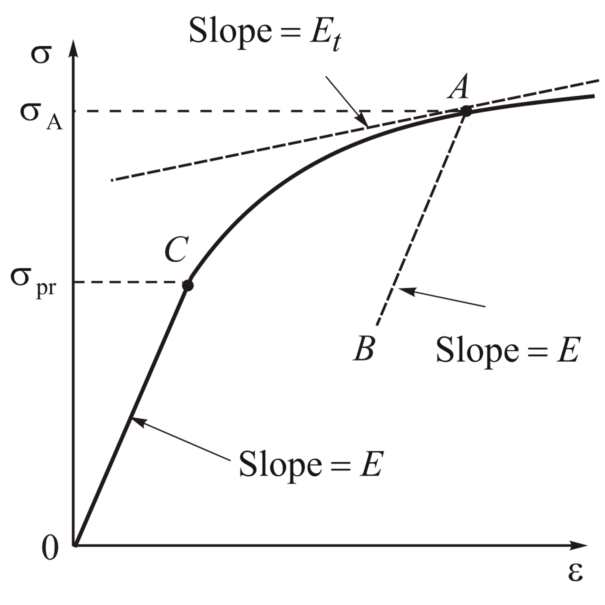


Fig. 3 Compression stress-strain diagram for the material of the column shown in Fig. 2

According to the tangent-modulus theory of inelastic buckling, the column shown in Fig. 2a remains straight until the inelastic critical load P_t is reached. As increments of load are imposed, the column displays a slight curvature, i.e. the column undergoes a small lateral deflection (Fig. 2b). The resulting bending stresses $\Delta\sigma$ are superimposed upon the axial compressive stresses $\sigma_A = \sigma_{cr}$, associated with the attainment of critical load P_t (Fig. 4). Since the column starts bending from a straight position, the initial bending stresses $\Delta\sigma$ represent only a small increment of stress. Therefore, the relationship between the bending stresses $\Delta\sigma$ and the change in strain $\Delta\varepsilon$ is therefore $\Delta\sigma = E_t \Delta\varepsilon$. Here E_t is tangent modulus of the material that is, $E_t = d\sigma / d\varepsilon$.

Since the strains vary linearly across the cross section of the column, the initial bending stresses also vary linearly, and therefore the expressions for curvature are the same as those for linearly elastic bending except that E_t replaces E:

$$k = \frac{1}{\rho} = \frac{d^2 v}{dx^2} = \frac{M}{E_t I}. \qquad (67)$$

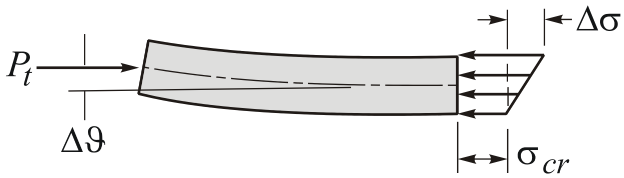


Fig. 4 Stress distribution in an intermediate column. The increment of stress $\Delta\sigma$ is due to bending; σ_{cr} represents the value of stress associated with the attainment of critical load P_t. The distribution of fibers strain will display the identical pattern.

Because the bending moment $M = -Pv$ (see Fig. 2b), the differential equation of the deflection curve is

$$E_t I v'' + Pv = 0 \qquad (68)$$

This equation has the same form as the equation for elastic buckling except that E_t appears in place of E. Therefore, we can solve the equation in the same manner as before and obtain the following equation for the **tangent-modulus load**:

$$P_t = \frac{\pi^2 E_t I}{L^2}. \qquad (69)$$

This load represents the critical load for the column according to the tangent-modulus theory. The corresponding critical stress may be expressed by the generalized Euler's buckling formula, or the tangent modulus formula:

$$\sigma_{cr} = \sigma_t = \frac{P_t}{A} = \frac{\pi^2 E_t}{\left(L/i\right)^2}. \qquad (70)$$

Because the tangent modulus E_t corresponding to the given stress in the compression stress-strain diagrams, we will obtain the tangent-modulus load P_t using in iterative procedure. We begin by estimating the value of P_t. This trial value, call it P_1 should be slightly larger than $\sigma_{pr}A$, which is the axial load when the stress just reaches the proportional limit. Knowing P_1, we can calculate the corresponding axial stress $\sigma_1 = P_1/A$ and determine the tangent modulus E_t from the stress-strain diagram. Next, we use Eq. 69 to obtain a second estimate of P_t, called as P_2. If P_2 is very close to P_1, we may accept the load P_2 as the tangent-modulus load. However, it is more likely that additional cycles of iteration will be required until we reach a load that is in close agreement with the preceding trial load. This value is resultant tangent-modulus load. Tangent-modulus critical stress σ_t range for intermediate columns is represented by the curve *BC* in Fig. 1.

It should be noted that ***Eq. (70) determines the ultimate stresses not the working stresses***. It is thus necessary to divide the right side of each formula by an appropriate factor of safety, often 2 to 3, depending upon the material, in order to obtain the allowable values.

A diagram showing how the critical stress σ_t varies with the slenderness ratio is given in Fig. 5 for a typical metal column with pinned ends. ***This curve is above the proportional limit and below Euler's curve***.

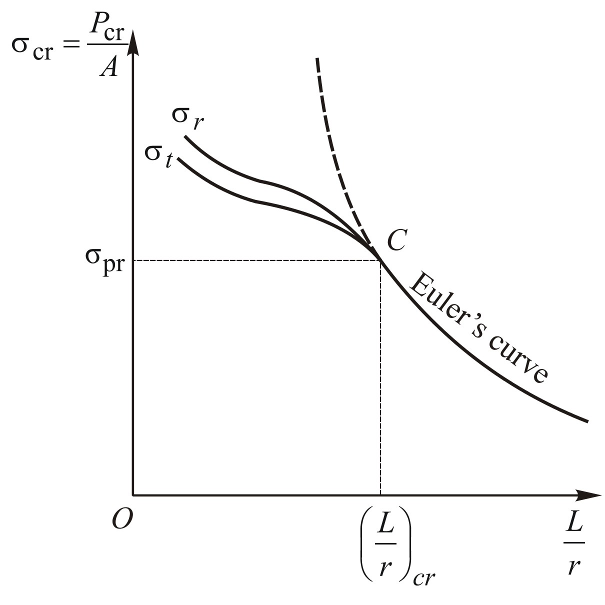


Fig. 5 Diagram of critical stress versus slenderness ratio

The tangent-modulus formulas may be used for columns with various support conditions by using the effective length L_e in place of the actual length L.

3 Reduced-Modulus Theory (Double-Modulus Theory)

The tangent-modulus theory is simple in use. But, it does not account for complete behavior of the column. Let us consider again the column shown in Fig. 2a. When this column first departs from the straight position (Fig. 2b), bending stresses are added to the existing compressive stresses P/A. These additional stresses are compressive on the concave side of the column and tensile on the convex side. Therefore, the compressive stresses in the column become larger on the concave side and smaller on the convex side.

Let us suppose that the axial stress P/A is represented by point A on the stress-strain curve (Fig. 3). On the concave side of the column (where the compressive stress is increased), the material follows the tangent modulus E_t. However, on the convex side (where the compressive stress is decreased), the material follows the unloading line *AB* the stress-strain diagram on Fig. 3. This line is parallel to the initial linear part of the diagram, and therefore its slope is equal to the elastic modulus E. Thus, at the onset of bending, the column behaves as if it were made of two different materials, ***a material of modulus E_t on the concave side and a material of modulus E on the convex side***.

A bending analysis of such a column can be made using the bending theories for a beam of two materials. The results of such analyses show that the column bends as though the material had a modulus of elasticity between the values of E and E_t. This "effective modulus" is known as the **reduced modulus** E_r, and ***its value depends not only upon the magnitude of the stress*** (because E_t depends upon the magnitude of the stress) ***but also upon the shape of the cross section of the column***. Thus, the reduced modulus E_r is more difficult to determine than is the tangent modulus E_t. In the case of a column having a rectangular cross section, the equation for the reduced modulus is

$$E_r = \frac{4EE_t}{\left(\sqrt{E}+\sqrt{E_t}\right)^2}. \tag{71}$$

For a wide-flange beam with the areas of the webs ignored, the reduced modulus for bending about the strong axis is

$$E_r = \frac{2EE_t}{E + E_t}. \tag{72}$$

Since the reduced modulus represents an effective modulus that governs the bending of the column when it first departs from the straight position, we can formulate a fundamental principles of reduced-modulus theory of inelastic buckling in the same manner as for the tangent-modulus theory. We will begin with an equation for the curvature and then we will write the differential equation of the deflection curve. These equations are the same as Eqs. (67) and (68) except that E_r appears instead of E_t. Thus, we arrive at the following equation for the **reduced-modulus load**:

$$P_r = \frac{\pi^2 E_r I}{L^2}. \tag{73}$$

The corresponding equation for the critical stress is

$$\sigma_r = \frac{\pi^2 E_r}{(L/i)^2} = \frac{\pi^2 E_r}{\lambda^2}. \tag{74}$$

To find the reduced-modulus load P_r, we again must use an iterative procedure, because E_r depends upon E_t. The critical stress according to the reduced-modulus theory is shown in Fig. 5. It is important to note that the ***curve for*** $\boldsymbol{\sigma_r}$ ***is above that for*** $\boldsymbol{\sigma_t}$***, because*** $\boldsymbol{E_r}$ ***is always greater than*** $\boldsymbol{E_t}$.

The reduced-modulus theory is difficult to use in practice because *Ошибка! Объект не может быть создан из кодов полей редактирования.* ***depends upon the shape of the cross section as well as the stress-strain curve and must be evaluated for each particular column.***

4 Shanley Theory

An understanding of the tangent-modulus and reduced-modulus theories is necessary in order to consider a more complete theory. It was developed by F.R. Shanley in 1946 and is called the **Shanley theory of inelastic buckling**.

The Shanley theory recognizes that it is not possible for a column to buckle inelastically in a manner that is analogous to Euler buckling. ***In Euler buckling, a critical load is reached at which the column is in neutral equilibrium***, represented by a horizontal line on the load-deflection diagram (Fig. 6). Neither the tangent-modulus load P_t nor the reduced-modulus load P_r can represent this type of behavior. In both cases, we are led to a contradiction if we try to associate the load with a condition of neutral equilibrium.

Instead of neutral equilibrium, wherein a deflected shape suddenly becomes possible with no change in load, we must think of a column that has an ever-increasing axial load. When the load reaches the tangent-modulus load (which is less than the reduced-modulus load), bending can begin only if the load continues to increase. Under these conditions, bending occurs simultaneously with an increase in load, resulting in a decrease in strain on the convex side of the column. Thus, the effective modulus of the material throughout the cross section becomes greater than E_t, and therefore an increase in load is possible. However, the effective modulus is not as great as E_r, because E_r is based upon full strain reversal on the convex side of the column. In other words, E_r is based upon the amount of strain reversal that exists if the column bends without a change in the axial force, whereas the presence of an increasing axial force means that the reduction in strain is not as great.

Thus, ***instead of neutral equilibrium***, where the relationship between load and deflection is undefined, ***we consider a definite relationship between each value of the load and the corresponding deflection***. This behavior is shown by the curve labeled "Shanley theory" in Fig. 6. Note that ***buckling begins at the tangent-modulus load; then the load increases but does not reach the reduced-modulus load until the deflection becomes infinitely large (theoretically)***. However, other effects become important as the deflection increases, and in reality the curve eventually goes downward, as shown by the dashed line.

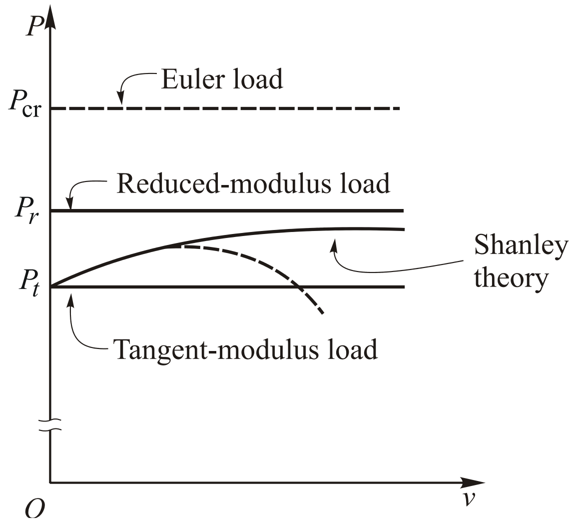
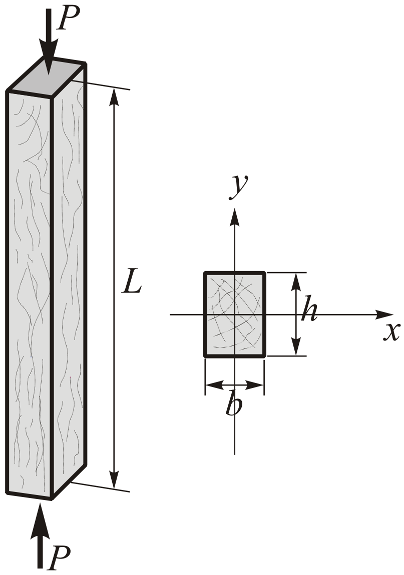


Fig. 6 Load-deflection diagram for elastic and inelastic buckling

The Shanley concept of inelastic buckling has been verified by numerous investigators and by many tests and is now fully accepted. However, the maximum load attained by real columns (see the dashed curve trending downward in Fig. 6) is only slightly above the tangent-modulus load P_t. In addition, the tangent-modulus load is very simple to calculate. Therefore, ***for practical purposes it is reasonable to adopt the tangent-modulus load as the critical load for inelastic buckling of columns***.

5 Examples



Example 1 A 2-meters-long Douglas fir bar of $b = 5\,\text{cm}$ by $h = 10\,\text{cm}$ rectangular cross section, pivoted at both ends, is subjected to an axial compressive load (see Figure). For $E = 12.5\,\text{GPa}$, calculate (1) the slenderness ratio and (2) the allowable load, using a factor of safety of $n = 1.5$.

Solution The minimum radius of gyration $i_{\min}$ of the cross section is

$$i_{\min} = i_y = \sqrt{\frac{I_{\min}}{A}} = \sqrt{\frac{hb^3}{12bh}} = \frac{b}{2\sqrt{3}}.$$

(1) The lowest value of i is obtained when the centroidal axis is parallel to the longer side of the rectangle, i.e. y-axis. We thus have $h = 10\,\text{cm}$ and $b = 5\,\text{cm}$ so that $i_{\min} = i_y = 5/2\sqrt{3} = 1.45\,\text{cm}$. Then maximum actual slenderness ratio $\lambda_{\max} = \lambda_y = L/i_{\min} = 2\times10^2/1.45\times10^{-2} = 138$.

(2) The Euler buckling load is

$$P_{cr} = \frac{\pi^2 EA}{\left(L/i_{\min}\right)^2} = \frac{\pi^2\left(12.5\times10^9\right)\left(5\times10^{-2}\right)\left(10\times10^{-2}\right)}{\left(138\right)^2} = 3.24\,\text{kN}.$$

The largest load the column can support is therefore $P_{\text{all}} = 3.24/1.5 = 2.16\,\text{kN}$.

Example 2 A long, slender column *ABC* is pin-supported at the ends and compressed by an axial load *P* (see Figure). Lateral support is provided at the midpoint *B* in the plane of the figure. However, lateral support perpendicular to the plane of the figure is provided only at the ends. The column is constructed of a steel wide-flange section $W8 \times 28$ (see Assortment of steel products) having modulus of elasticity $E = 200\,\text{GPa}$ and proportional limit $\sigma_{pr} = 290\,\text{MPa}$. The total length of the column is $L = 7.6\,\text{m}$.

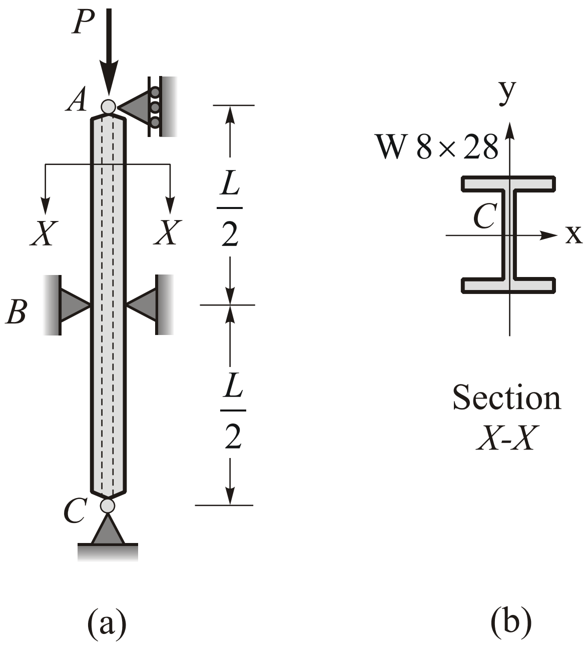


Figure. Euler's buckling of a slender column

Determine the allowable load P_{all} using a factor of safety $n = 2.0$ with respect to Euler buckling of the column.

Solution Because of the manner in which it is supported, this column may buckle in either of the two principal planes of bending. First, it may buckle in the plane of the figure, in which case the distance between lateral supports is $L/2 = 3.8\,\text{m}$ and bending occurs about axis *y*. Second, it may buckle perpendicular to the plane of the figure with bending about axis *x*. Because the only lateral support in this direction is at the ends, the distance between lateral supports is $L = 7.6\,\text{m}$.

Column properties. From Assortment of steel products, we obtain the following moments of inertia and cross-sectional area for a $W8 \times 28$ column:

$$I_x = 98.0\ \text{in.}^4 = 4079 \times 10^{-8}\ \text{m}^4;\ I_y = 21.7\ \text{in.}^4 = 903 \times 10^{-8}\ \text{m}^4,$$

$$A = 8.25\ \text{in.}^2 = 53.23 \times 10^{-4}\ \text{m}^2.$$

Checking the Euler's formula applicability. As previously mentioned, the Euler's curve is valid only when critical stress is less than the proportional limit of the material, because the equations were derived using Hooke's law. It means, that the Euler's curve is limited by the horizontal line at the proportional limit of the steel ($\sigma_{pr} = 290\,\text{MPa}$), and an applicability of the Euler's formula must be checked after finding the critical stress.

The same checking is in comparison of limiting and actual slenderness ratios. In this example, limiting slenderness ratio $\lambda_{\lim} = \pi\sqrt{E/\sigma_{pr}} = \pi\sqrt{2\times10^{11}/290\times10^{6}} = 82.6$. Actual maximum slenderness ratio $\lambda_{\max} = L/i_{\min} = L/i_2 = 7.6/(1.62)\times10^{-2} = 469$. Because $\lambda_{\max} > \lambda_{\min}$ the Euler's formula is applicable.

Critical loads. If the column buckles in the plane of the figure, the critical load is

$$P_{cr} = \frac{\pi^2 EI_y}{(L/2)^2} = \frac{4\pi^2 EI_y}{L^2} = \frac{4\pi^2\left(2\times10^{11}\ \text{Pa}\right)\left(903\times10^{-8}\ \text{m}^4\right)}{(7.6\ \text{m})^2} = 1233\,\text{kN}.$$

If the column buckles perpendicular to the plane of the figure, the critical load is

$$P_{cr} = \frac{\pi^2 EI_x}{L^2} = \frac{\pi^2\left(2\times10^{11}\ \text{Pa}\right)\left(4079\times10^{-8}\ \text{m}^4\right)}{(7.6\ \text{m})^2} = 1393\,\text{kN}.$$

Therefore, the critical load for the column (the smaller of the two preceding values) is:

$$P_{cr} = 1233\,\text{kN}$$

and buckling occurs in the plane of the figure.

Critical stresses. Since the calculations for the critical loads are valid only if the material follows Hooke's law, we need to verify that the critical stresses do not exceed the proportional limit of the material. In the case of the larger critical load, we obtain the following critical stress:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{1393\times10^{3}\ \text{N}}{53.23\times10^{-4}\ \text{m}^2} = 267\,\text{MPa}.$$

Since this stress is less than the proportional limit ($\sigma_{\text{pr}} = 290\,\text{MPa}$), both critical load calculations are valid.

Allowable load. The allowable axial load for the column, based on Euler buckling and the factor of safety $n = 2$, is

$$P_{all} = \frac{P_{cr}}{n} = \frac{1233\,\text{kN}}{2.0} = 616.5\,\text{kN}$$

in which $n = 2.0$ is the factor of safety.

Example 3 A 76 by 64 by 6.4-mm steel angle (see Assortment of steel products) (see Figure) is to serve as a pin-ended column to support a 40 kN load with a safety factor $n = 2$. Assuming that the proportional limit $\sigma_{pr} = 240\,\text{MPa}$ and $E = 210\,\text{GPa}$, determine the maximum length of the member.

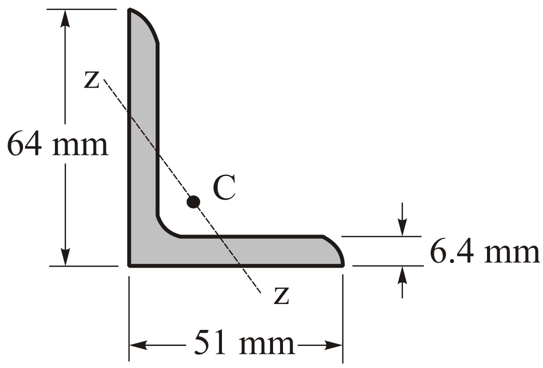


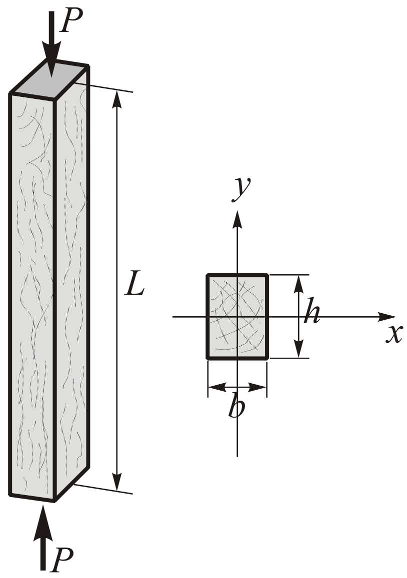
Figure. Nonequileg cross-section with *z-z* minimum principal axis of inertia

Solution From (see Assortment of steel products) we find that the smallest radius of gyration for the principal axis (z) of the angle cross section is $i_{\min} = 13.4\,\text{mm}$; the area is $850\,\text{mm}^2$. In this case, the critical load becomes $P_{cr} = 2(40) = 80\,\text{kN}$. Thus $\sigma_{cr} = 80 \times 10^3 / (850 \times 10^{-6}) = 94.1\,\text{MPa}$ and we have $\sigma_{cr} < \sigma_{pr}$. Based upon the assumption that Euler's formula is applicable, we have

$$\sigma_{cr} = \frac{\pi^2 E}{\left(L / i_{\min}\right)^2} \quad \text{or} \quad 94.1 \times 10^6 = \frac{\pi^2 \left(210 \times 10^9\right)}{\left(L / 0.0134\right)^2},$$

from which $L = 1.25\,\text{m}$. Maximum actual slenderness ratio $\lambda_{\max} = L / i_{\min} = 1.25/0.0134 = 93.3$, is well into the Euler range, the assumption is valid.

Example 4 Calculate the critical load of a wooden past ($E = 10\,\text{GPa}$, $\sigma_{pr} = 8\,\text{MPa}$) of 3 by 20 mm rectangular cross section and the length of 0.3 m. Estimate the change in critical load if the length of the post is decreased: 1) two times; 2) four times. Assume that the supports provide pinned-end conditions.



Solution *Past properties*. The minimum radius of gyration $i_{\min}$ of the cross section is

$i_{\min} = i_y = \sqrt{\dfrac{I_y}{A}} = \sqrt{\dfrac{hb^3}{12hb}} = \dfrac{b}{2\sqrt{3}} = 0.867\,\text{mm}.$ Then, maximum actual slenderness ratio $\lambda_{\max} = \lambda_y = L/i_y = 0.3\Big/\left(0.867\times10^{-3}\right) = 346$.

Checking the Euler's formula applicability for given yardstick.

Limiting slenderness ratio for wood $\lambda_{\lim} = \pi\sqrt{E/\sigma_{pr}} = \pi\sqrt{\left(10\times10^9\right)\left(8\times10^6\right)} = 111$.

Because $\lambda_{\max} > \lambda_{\lim}\ (346 > 111)$, the Euler's formula is applicable.

Critical load for given length $L = 0.3\,\text{m}$.

$$P_{cr} = \frac{\pi^2 E I_{\min}}{L^2} = \frac{\pi^2 E I_z}{L^2} = \frac{\pi^2 E h b^3}{12L^2} =$$

$$= \frac{\pi^2\left(0.1\times10^{11}\text{Pa}\right)\left(20\times10^{-3}\text{m}\right)\left(3\times10^{-3}\text{m}\right)^3}{12\left(0.3\text{m}\right)} = 14.8\,\text{N}.$$

Checking the Euler's formula applicability for the column of twice decreased length $L_1 = L/2 = 0.15\,\text{m}$.

Evidently, limiting slenderness ratio remain unchanged: $\lambda_{\lim} = 111$. Maximum actual slenderness ratio becomes $\lambda_{\max_1} = \lambda_y = L_1/i_y = 0.15\Big/\left(0.867\times10^{-3}\right) = 173$. Because $\lambda_{\max 1} > \lambda_{\lim}\ (173 > 111)$, the Euler's formula remains applicable. Critical load for the yardstick with twice decreased length $L_1 = 0.15\,\text{m}$, $P_{cr1} = 4P_{cr} = 59.2\,\text{N}$.

Checking the Euler's formula applicability for the column of fourth decreased length $L_2 = L/4 = 0.075\,\text{m}$.

Limiting slenderness ratio remains unchanged: $\lambda_{\lim} = 111$. Maximum actual slenderness ratio becomes $\lambda_{\max 2} = \lambda_y = L_2/i_y = 0.075\Big/\left(0.867\times10^{-3}\right) = 86.5$.

Because $\lambda_{\max 2} < \lambda_{\lim}\,(86.5 < 111)$, the Euler's formula becomes inapplicable and to calculate critical load for this intermediate column it is necessary to apply nonelastic analysis.

Example 5 A 102 by 76 by 12.7-mm steel angle (see Assortment of steel products) is used as a pin-ended column to support 60 kN load. Assuming that the proportional limit $\sigma_{pr} = 240\,\text{MPa}$ and $E = 210\,\text{GPa}$ calculate the critical stress σ_{cr}, and the critical load P_{cr} if the length $L = 2\,\text{m}$.

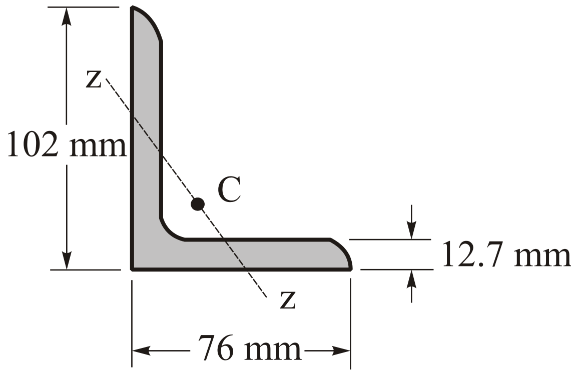


Figure. Nonequileg cross-section with *z-z* minimum principal axis of inertia

Solution *The column properties.* From see Assortment of steel products, we find that the smallest radius of gyration for the principal axis (z) of this angle cross section is $i_{\min} = 16.2\,\text{mm}$; the area is 2100 mm^2. Maximum actual slenderness ratio

$$\lambda_{\max} = \lambda_z = L/i_z = 2\Big/\left(16.2\times10^{-3}\right) = 123.5.$$

Checking the Euler's formula applicability.

Limiting slenderness ratio for the steel

$$\lambda_{\lim} = \pi\sqrt{E/\sigma_{pr}} = \pi\sqrt{\left(210\times10^{9}\right)\Big/\left(240\times10^{6}\right)} = 91.7.$$

Because $\lambda_{\max} > \lambda_{\lim}\,(123.5 > 91.7)$, the Euler's formula is applicable.

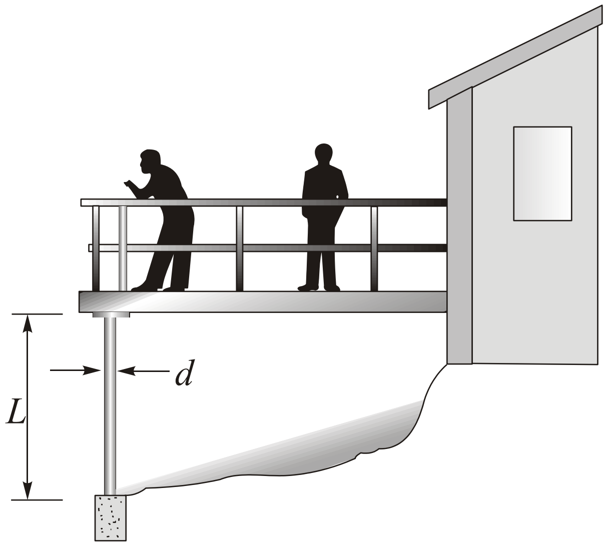
Critical stress for given column.

$$\sigma_{cr} = \frac{\pi^2 E}{\lambda^2} = \frac{\pi^2\left(210\times10^9\right)}{123.5^2} = 135.8\,\text{MPa}.$$

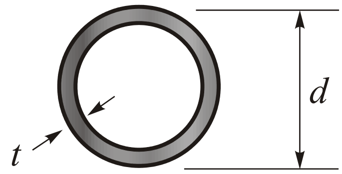
Evidently, that because buckling is within elasticity limits, $\sigma_{cr} < \sigma_{pr}$ $(135.8 < 240)$.

Critical load. $P_{cr} = \sigma_{cr}A = \left(135.8\times10^6\,\text{Pa}\right)\left(2100\times10^{-6}\,\text{m}^2\right) = 285.2\,\text{kN}.$

Example 6 The platform (see Figure) is supported by a row of aluminum pipe columns having length $L = 3.25\,\text{m}$ and outer diameter $d = 100\,\text{mm}$. The bases of the columns are fixed in concrete footings and the tops of the columns are supported laterally by the platform. The columns are being designed to support compressive loads $P = 100\,\text{kN}$. Determine the minimum required thickness t of the columns, if a factor of safety $n = 3$ is required with respect to Euler buckling. Use for the aluminum, $E = 72\,\text{GPa}$ for the modulus of elasticity and use $\sigma_{pr} = 480\,\text{MPa}$ for the proportional limit.



(a)



(b)

Figure. A platform supported by pipe columns

Solution Because of the manner in which the columns are constructed, we will simulate each column as a fixed-pinned column (see Figure). Therefore, the critical load is

$$P_{cr} = \frac{2.046\pi^2 EI}{L^2}, \qquad \text{(a)}$$

in which I is the moment of inertia of the tubular cross section:

$$I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]. \qquad \text{(b)}$$

Substituting $d = 100\,\text{mm}$ (or 0.1 m), we get

$$I = \frac{\pi}{64}\left[(0.1\text{ m})^4 - (0.1\text{ m} - 2t)^4\right], \tag{c}$$

in which t is expressed in meters.

Calculation of thickness of the columns. Since the load per column is 100 kN and the factor of safety is 3, each column must be designed for the following critical load:

$$P_{cr} = nP = 3(100\text{ kN}) = 300\,\text{kN}.$$

Substituting this value for P_{cr} in Eq. (a), and also replacing I with its expression from Eq. (c), we obtain

$$300{,}000\text{ N} = \frac{2.046\pi^2\left(72\times10^9\,\text{Pa}\right)}{(3.25\text{ m})^2}\left(\frac{\pi}{64}\right)\left[(0.1\text{ m})^4 - (0.1\,\text{m} - 2t)^4\right].$$

After multiplying and dividing, the preceding equation simplifies to

$$44.40\times10^{-6}\text{ m}^4 = (0.1\text{ m})^4 - (0.1\text{ m} - 2t)^4,$$

or

$$(0.1\text{ m} - 2t)^4 = (0.1\text{ m})^4 - 44.40\times10^{-6}\text{ m}^4 = 55.60\times10^{-6}\text{ m}^4,$$

from which we obtain

$$0.1\text{ m} - 2t = 0.08635\,\text{m} \quad \text{and} \quad t = 0.006825\,\text{m}.$$

Therefore, the minimum required thickness of the column to meet the specified conditions is

$$t_{\min} = 6.83\,\text{mm}.$$

Note, that Euler's formula was used in the calculation unsubstantially. That's why, knowing the diameter and thickness of the column, we must now calculate its moment of inertia, cross-sectional area, and radius of gyration to find actual slenderness ratio and compare it with limiting slenderness ratio of column material. Using the minimum thickness of 6.83 mm, we obtain

$$I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right] = 2.18\times10^6\,\text{mm}^4,$$

$$A = \frac{\pi}{4}\left[d^2 - (d-2t)^2\right] = 1999\,\text{mm2}, \quad i = \sqrt{\frac{I}{A}} = 33.0\,\text{mm}.$$

Due to the assumption of pinned supports, the actual slenderness ratio $\lambda = L/i$ of the column is approximately 98. Critical slenderness ratio $\lambda_{cr} = \pi\sqrt{E/\sigma_{pr}} = 3.14\sqrt{72\times10^9/\left(480\times10^6\right)} = 38.3$. Because $\lambda > \lambda_{cr}$ $(98 > 38.3)$ the application of the Euler's formula is valid.

At the same time the critical stress in the column must be less than the proportional limit of the aluminum if the formula for the critical load (Eq. a) is to be valid. The critical stress is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{300\text{ kN}}{1999\text{ mm}^2} = 150\,\text{MPa},$$

which is less than the proportional limit (480 MPa). Therefore, our calculation for the critical load using Euler buckling theory is satisfactory.

Example 7 Rigidly fixed and cable-supported rectangular steel bar *AB* in the structure (see Figure), is constructed of a 50-mm by 75-mm section, for which $E = 207\,\text{GPa}$ and $\sigma_{pr} = 250\,\text{MPa}$. What is the buckling load of the steel bar *AB*, if its length is 3.0 m.

Solution From the equations of static equilibrium, the axial compressed force expressed in terms of *W* is found to be $P = 4W/3$.

Geometrical properties of the cross section:

$$I_y = \frac{1}{12}(75)(50)^3 = 781250\,\text{mm}^4,$$

$$I_z = \frac{1}{12}(50)(75)^3 = 175782\,\text{mm}^4,$$

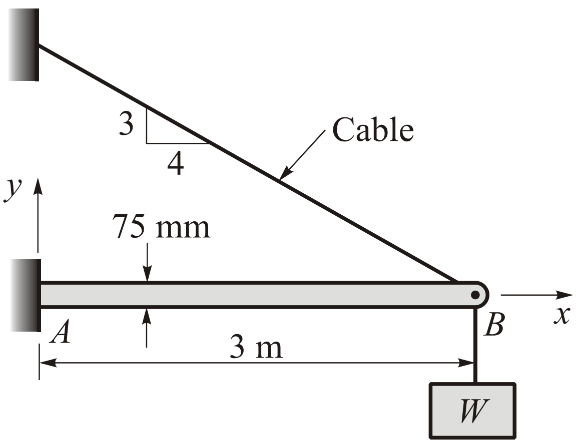


Figure. Cable-supported rectangular steel bar

$$A = (50)(75) = 3500\,\text{mm}^2,$$

and

$$i_y = \sqrt{\frac{I_y}{A}} = 14.94\,\text{mm}, \quad i_z = \sqrt{\frac{I_z}{A}} = 22.41\,\text{mm}.$$

Buckling in vertical xy plane. Due to the assumption of pinned support in *B* point, the effective length of the column with respect to buckling in vertical *xy* plane is $L_e = 0.7L$, therefore

$$\lambda_z = \frac{L_e}{i_z} = \frac{0.7 \times 3.0}{22.41 \times 10^{-3}} = 93.7.$$

The Euler formula is applicable in this range, because

$$\lambda_{cr} = \pi\sqrt{E/\sigma_{pr}} = \pi\sqrt{207 \times 10^9 / 250 \times 10^6} = 90.35, \quad \text{i.e.} \quad \lambda_z > \lambda_{cr} \quad (93.7 > 90.35).$$

Hence

$$P_{cr} = \frac{\pi^2 EA}{(\lambda_z)^2} = \frac{\pi^2\left(207 \times 10^9\right)\left(3500 \times 10^{-6}\right)}{(93.7)^2} = 813.6\text{ kN} = \frac{4}{3}W.$$

Using this result, $W = 3P_{cr}/4 = 3 \times 813.6/4 = 610.2\,\text{kN}$.

Simultaneously, $\sigma_{cr} = P_{cr}/A = \left(813.6 \times 10^3\right) / \left(3500 \times 10^{-6}\right) = 232.5\,\text{MPa}$.

The result is within the Euler's formula limitations, because $\sigma_{cr} < \sigma_{pr}$ $(232.5 < 250)$.

The Euler's formula is valid.

Buckling in horizontal xz plane. There are no any supports in point *B* in *xz* plane. Simultaneously, point *A* remains rigidly fixed. Therefore **Ошибка! Объект не может быть создан из кодов полей редактирования.**. Then

$$\lambda_y = \frac{L_e}{i_y} = \frac{2 \times 3.0}{14.94 \times 10^{-3}} = 401.6.$$

As previously, $\lambda_y > \lambda_{cr}$ $(401.6 > 90.35)$.

The buckling load is

$$P_{cr}=\frac{\pi^2 EA}{\left(\lambda_y\right)^2}=\frac{\pi^2\left(207\times10^9\right)\left(3500\times10^{-6}\right)}{\left(401.6\right)^2}=44.3\,\text{kN}=\frac{4}{3}W\,.$$

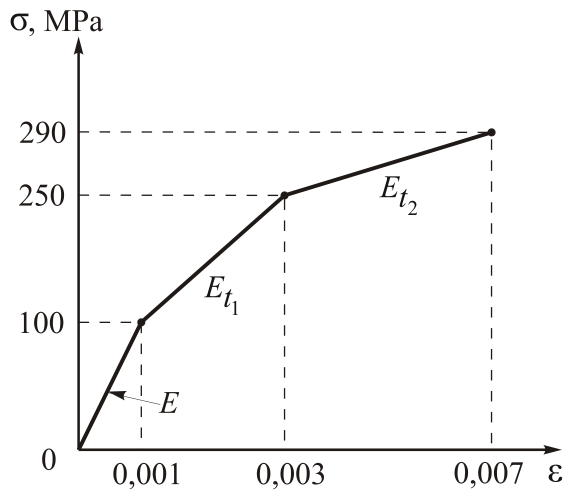
From this $W=3P_{cr}/4=3\times44.3/4=33.2\,\text{kN}$.

Simultaneously $\sigma_{cr}=P_{cr}/A=\left(44.3\times10^3\right)\Big/\left(3500\times10^{-6}\right)=12.7\,\text{MPa}$.

As previously, the result is within the Euler's formula limitations, because $\sigma_{cr}<\sigma_{pr}\quad(12.7<250)$.

The column will obviously fail by lateral buckling when load W exceeds 33.2 kN. Observe that the critical stress is only equal to 12.7 MPa. This compared with the proportional limit 250 MPa demonstrates the significance of buckling analysis in predicting the safe working load.

Example 8 A pipe of 76-mm outside diameter and 3-mm thickness is used as a column of 2-m effective length. Determine the axial buckling load for a material with a stress-strain curve approximated as in the figure.



Solution For a tube cross section, geometric properties are

$$A=\frac{\pi}{4}\left(76^2-70^2\right)=688\,\text{mm}^2,$$

$$i=\frac{1}{4}\sqrt{76^2+70^2}=25.8\,\text{mm}.$$

Thus, actual slenderness ratio $\lambda=L/i=77.5$ and the tangent modulus formula is

$$\sigma_{cr}=\frac{\pi^2E_t}{\left(L_e/i\right)^2}=\frac{\pi^2E_t}{\left(77.5\right)^2}=0.001643E_t\,.$$

Because the value of E_t is different in every range, we must employ a trial-and-error procedure.

For the initial (elastic) range, the slope is $E_t=E=100\,\text{GPa}$ and the last equation yields

$$\sigma_{cr} = 0.001643\left(100\times10^{9}\right) = 164.3\,\text{MPa}.$$

As observed from the Figure, E is not valid above 100 MPa. For the second portion of the stress-strain curve, $E_{t1} = 75\,\text{GPa}$ and therefore

$$\sigma_{cr} = 0.001643\left(75\times10^{9}\right) = 123.2\,\text{MPa}.$$

This is the critical stress at buckling because the value of E_{t1} lies between 100 and 250 MPa. The solution is therefore

$$P_{cr} = \sigma_{cr}A = \left(123.2\times10^{6}\right)\left(688\times10^{-6}\right) = 84.8\,\text{kN}.$$

Example 9 Calculate the critical load for the pin-ended aluminum pipe column ($E = 70\,\text{GPa}$ for the modulus of elasticity and $\sigma_{pr} = 250\,\text{MPa}$ for the proportional limit) having length $L = 1.5$ and outer diameter $d_1 = 100\,\text{mm}$ and inner diameter $d_2 = 80\,\text{mm}$.

Solution 1. *Column properties*.

For a tube cross section, we have

$$A = \frac{\pi}{4}\left(d_1^2 - d_2^2\right) = \frac{\pi}{4}\left(100^2 - 80^2\right) = 2826\,\text{mm}^2,$$

$$i = \sqrt{\frac{I}{A}} = \frac{1}{4}\sqrt{\frac{\left(d_1^4 - d_2^4\right)}{\left(d_1^2 - d_2^2\right)}} = \frac{1}{4}\sqrt{d_1^2 + d_2^2} = \sqrt{\frac{1}{4}}\sqrt{100^2 + 80^2} = 32.0\,\text{mm}.$$

The actual slenderness ratio $\lambda = L/i = 1.5\Big/\left(32.0\times10^{-3}\right) = 46.9$.

2. *Limiting slenderness ratio for given aluminum alloy.*

$$\lambda_{\lim} = \pi\sqrt{E/\sigma_{pr}} = 3.14\sqrt{\left(70\times10^{9}\right)\Big/\left(250\times10^{6}\right)} = 52.5.$$

3. *Checking the Euler's formula applicability.*

Evidently, $\lambda < \lambda_{\min}\left(46.9 < 52.5\right)$, i.e. the Euler's formula is inapplicable, and to calculate the critical load for this intermediate column it is necessary to apply non-elastic analysis, in particular, the Jasinsky formula.

4. *Non-elastic analysis of the tube buckling using Jasinsky formula* $P_{cr} = A\left(a - b\lambda\right)$.

From the Table 1 $a = 398$ MPa, **Ошибка! Объект не может быть создан из кодов полей редактирования.**MPa, and

$P_{cr} = \left(2826 \times 10^{-6}\right)\left(398 - 2.78 \times 46.9\right)10^{6} = 756.2$ kN.

5. *Discussion of result.*

For the column under consideration critical stresses

$$\sigma_{cr} = \left(a - b\lambda\right) = \left(398 - 2.78 \times 46.9\right)10^{\,6} = 267.6 \text{ MPa}.$$

Evidently, $\sigma_{cr} > \sigma_{pr}\left(267.6 > 250\right)$, and application of the Euler's formula is incorrect. It gives

$$P_{cr} = \frac{\pi^2 EA}{\lambda^2} = \frac{\pi^2\left(70 \times 10^9\right)\left(2826 \times 10^{-6}\right)}{46.9^2} = 886.7 \text{ kN}.$$

It is important to note, that $756.2 < 886.7$ kN. It means that actual critical force is less than the one predicted by the Euler's formula. The history of engineering knows many fractures of machines and structures caused by excess critical stresses not predicted by elastic analysis and by application of Euler's formula in solution of buckling problem.