LECTURE 27 Stresses in Symmetrical Shells

1 Characteristic features of shells

Most elements of engineering structures to be designed may be reduced to the design schemes of a rod or a shell.

As previously mentioned, by a rod is meant any body one of whose dimensions (length) is considerably greater than the other two.

By a shell is meant a body one of whose dimensions (thickness) is considerably less than the other two. The examples of shells are shown in Fig. 1.

Fig. 1

The locus of points equidistant from both surfaces of a shell is called the **middle surface**. If the middle surface of a shell is a plane, such a shell is called a **plate**.

A shell is thin-walled if

$$\frac{\delta}{R_{\min}} < \frac{1}{10}, \qquad (1)$$

where δ is the **thickness** of a shell, $R_{\min}$ is the **radius**.

Symmetrical shells are those whose middle surface represents a surface of revolution.

It will be assumed that the load acting on such a shell also possesses the properties of axial symmetry. The analysis of such shells is greatly simplified. The stress-analysis problem for shell of revolution is simplest to solve when it can be assumed that the stresses developed in the shell are uniformly distributed through its thickness, and so the shell undergoes no bending.

2 Determination of stresses in symmetrical shells. Laplace's equation

Consider a symmetrical shell of thickness δ:

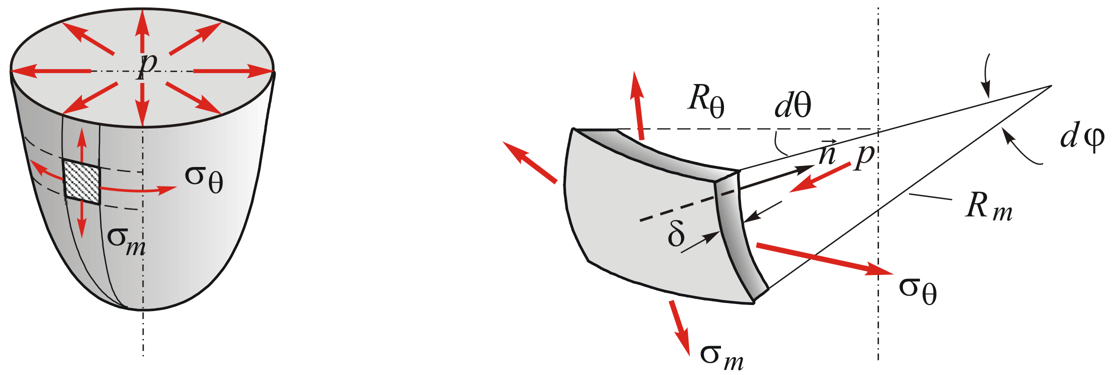

Fig. 2

Let R_m denote the radius of curvature of the meridian arc of the middle surface and R_θ the second principal radius, i.e. the radius of curvature of the normal section perpendicular to the meridian arc.

By two pairs of meridional and normal conical sections (Fig. 2) we isolate from the shell the element dS_1, dS_2 represented in Fig. 2. We shall assume that the faces of the element are acted on by stresses σ_m and σ_θ. The first stress σ_m is called the **meridional stress**, the vector of this stress is directed along the meridian arc. The second stress σ_θ is called the **circumferential stress**.

The stresses σ_m and σ_θ multiplied by the appropriate areas of the faces of the element give the forces $\sigma_m \delta dS_2$ and $\sigma_\theta \delta dS_1$ shown in Fig. 3.

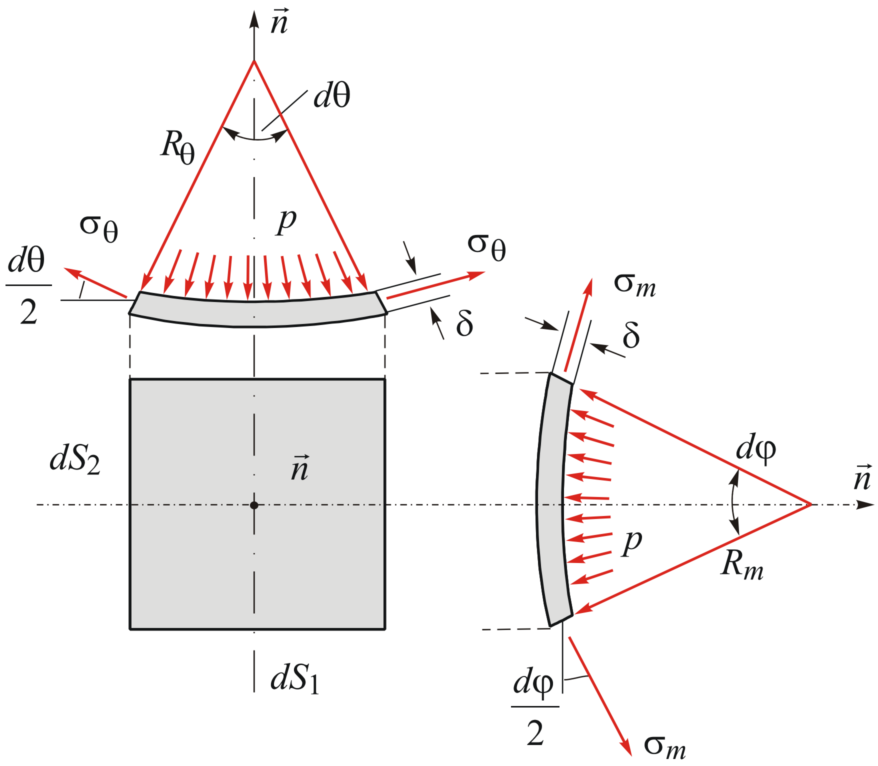

Fig. 3

The element is also subjected to the external normal pressure p. By projecting all forces on the normal, we obtain

$$pdS_1dS_2 - 2\sigma_m \delta dS_2 \sin\frac{d\varphi}{2} - 2\sigma_\theta \delta dS_1 \sin\frac{d\theta}{2} = 0,$$

since

$$\sin\frac{d\varphi}{2} \approx \frac{d\varphi}{2}, \quad \sin\frac{d\theta}{2} \approx \frac{d\theta}{2},$$

$$pdS_1dS_2 - \sigma_m \delta dS_2 \sin d\varphi - \sigma_\theta \delta dS_1 \sin d\theta = 0,$$

since

$$dS_2 = R_\theta d\theta \to d\theta = dS_2 / R_\theta,$$
$$dS_1 = R_m d\varphi \to d\varphi = dS_1 / R_m,$$

we have

$$pdS_1dS_2 - \sigma_m \delta dS_2 \frac{dS_1}{R_m} - \sigma_\theta \delta dS_1 \frac{dS_2}{R_\theta} = 0,$$

consequently

$$\frac{\sigma_m}{R_m} + \frac{\sigma_\theta}{R_\theta} = \frac{p}{\delta}. \tag{2}$$

This relation is known as **Laplace's equation**.

For the element shown in Fig. 2 it is possible to derive one more equation by projecting all forces on the direction of the axis of the shell. It is more convenient, however, to do this for a part of the shell cut off by a conical normal section. Hence we can determine the meridional stress σ_m. Thus, the stresses σ_m and σ_θ in a shell are determined from the equations of equilibrium.

The third principal stress, the stress of pressure between the layers of the shell, is assumed to be small and the state of stress of the shell is considered to be biaxial.

Example 1 A spherical shell of radius R and thickness δ is subjected to internal pressure p (Fig. 4). Determine the stresses developed in the shell.

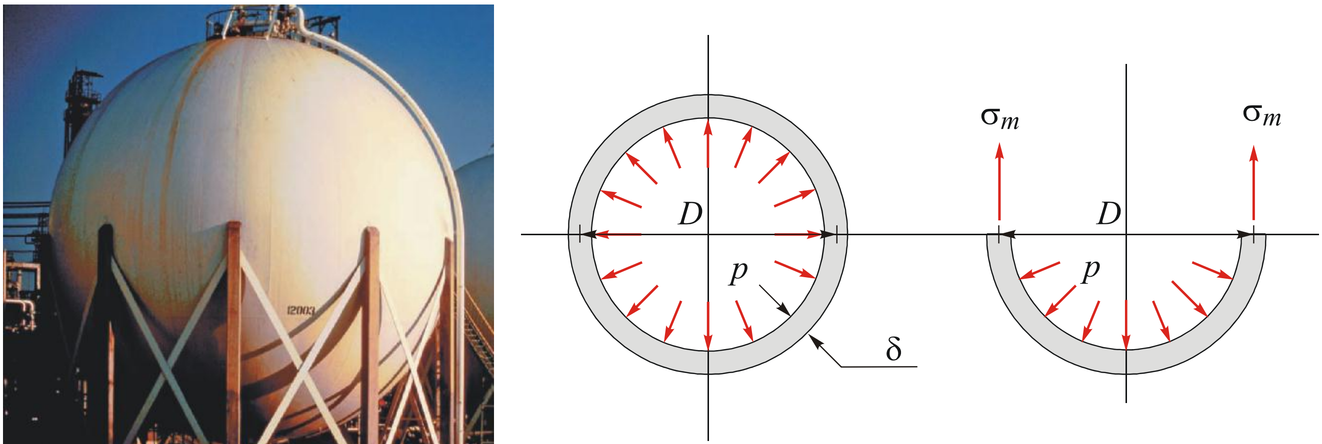

Fig. 4

For a spherical shell

$$R_m = R_\theta = R.$$

Because of complete symmetry

$$\sigma_m = \sigma_\theta.$$

Laplace's formula (2) gives

$$\sigma_m = \sigma_\theta = \frac{pR}{2\delta}.$$

The state of stress is biaxial

$$\sigma_1 = \sigma_2 = \frac{pR}{2\delta}.$$

The minimum stress σ_3 is assumed to be zero. By the third strength theory, condition of strength is

$$\sigma_{eq}^{III} = \frac{pR}{2\delta} \le [\sigma].$$

Example 2 A cylindrical vessel is subjected to internal pressure p. The radius of the cylinder is R, its thickness is δ. Determine the stresses.

We cut off part of the cylinder by a transverse section (Fig. 5) and derive an equation of equilibrium for it

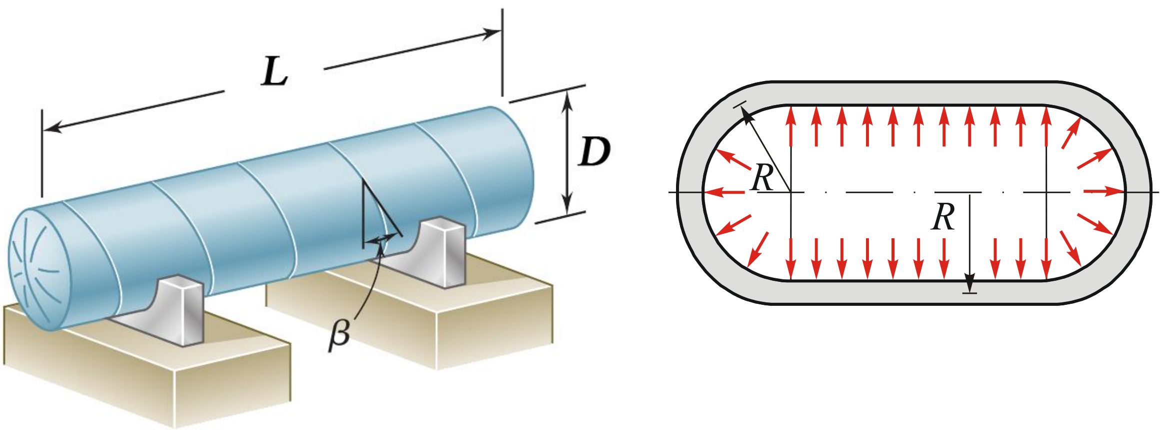

Fig. 5

$$2\pi R\delta\sigma_m = \pi R^2 p \to \sigma_m = \frac{pR}{2\delta}.$$

For a cylinder $R_m = \infty$, $R_\theta = R$. Hence from Laplace's formula we find

$$\frac{\sigma_m}{R_m} + \frac{\sigma_\theta}{R_\theta} = \frac{p}{\delta} \to \sigma_\theta = \frac{pR}{\delta},$$

i.e. ***the circumferential stress is twice the meridional stress***.

The element A isolated from the cylindrical shell is in the state of biaxial stress

$$\sigma_1 = \sigma_\theta,$$
$$\sigma_2 = \sigma_m,$$
$$\sigma_3 = -p = 0.$$

The equivalent stress is

$$\sigma_{eq}^{III} = \sigma_1 - \sigma_3 = \frac{pR}{\delta} \le [\sigma].$$

Example 3 A conical vessel of thickness δ is filled with a fluid of specific weight γ. Determine the stresses in the vessel (Fig. 6).

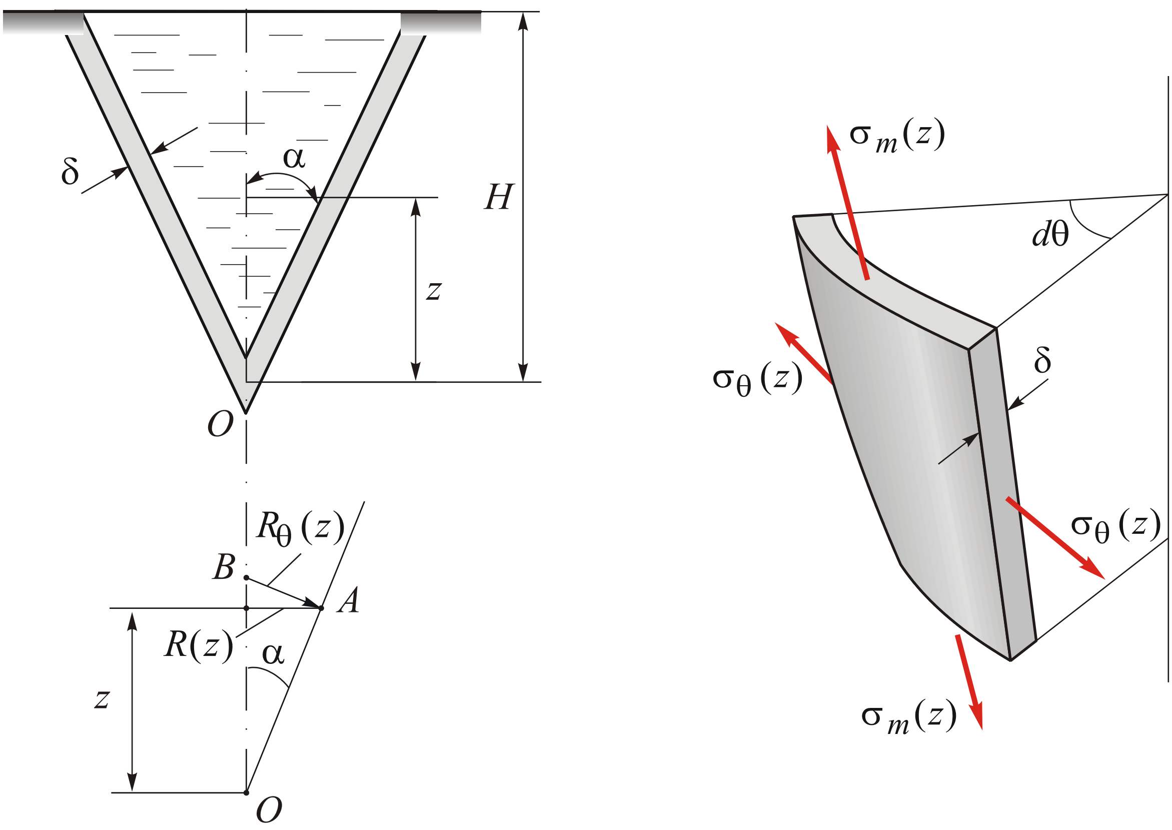

Fig. 6

We cut off a lower portion of the conical shell by a normal conical section. It is well known, that the pressure is equal to the weight of the fluid in the volume over the cut off portion of the shell:

$$p(z) = (H - z)\gamma .$$

The radius $R_m = \infty$;

$$OA = \frac{z}{\cos\alpha}; \quad R_\theta(z) = \frac{z}{\cos\alpha} tg\alpha .$$

By Laplace's formula

$$\frac{\sigma_m}{R_m} + \frac{\sigma_\theta}{R_\theta} = \frac{p}{\delta} \to \qquad \sigma_\theta(z) = \frac{p(z)R_\theta(z)}{\delta} = \frac{(H-z)}{\delta}\gamma\frac{z}{\cos\alpha}tg\alpha ,$$

$$\sigma_\theta(z) = \frac{\gamma tg\alpha}{\delta\cos\alpha}(Hz - z^2) .$$

Due to this function is parabolic, it is necessary to determine its extremal value:

$$\frac{d\sigma_\theta}{dz} = 0, \quad \frac{\gamma tg\alpha}{\delta\cos\alpha}\left(H - 2z_e\right) = 0 \to \qquad z_e = \frac{H}{2} .$$

Maximum stress $\sigma_{\theta_{\max}}$ occurs at inner points of the conical shell at $z = \frac{H}{2}$. The graph of this stress distribution is shown on Fig. 8 (left). $\sigma_{\theta_{\max}}$ value is determined by the formula:

$$\sigma_{\theta_{\max}} = \frac{1}{4}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2 .$$

Let us determine $\sigma_m(z)$ by projecting all forces on vertical z axes taking into account that the weight of situated below conical part is:

$$V_c(z) = \frac{1}{3}\pi R^2(z)z \quad \text{and} \quad R(z) = ztg\alpha .$$

Corresponding part of the shell is shown on Fig. 7. The equation of equilibrium is

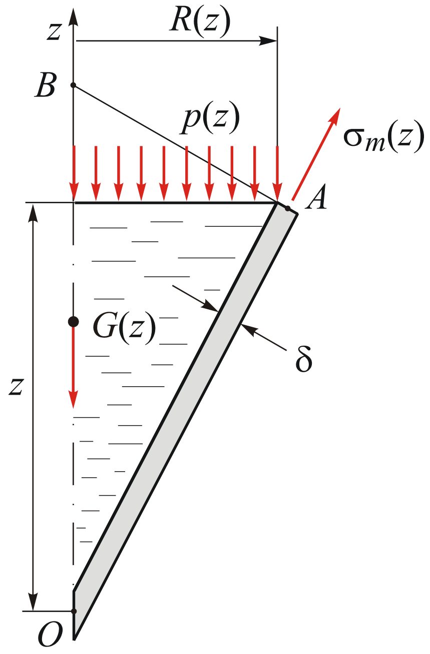

Fig. 7

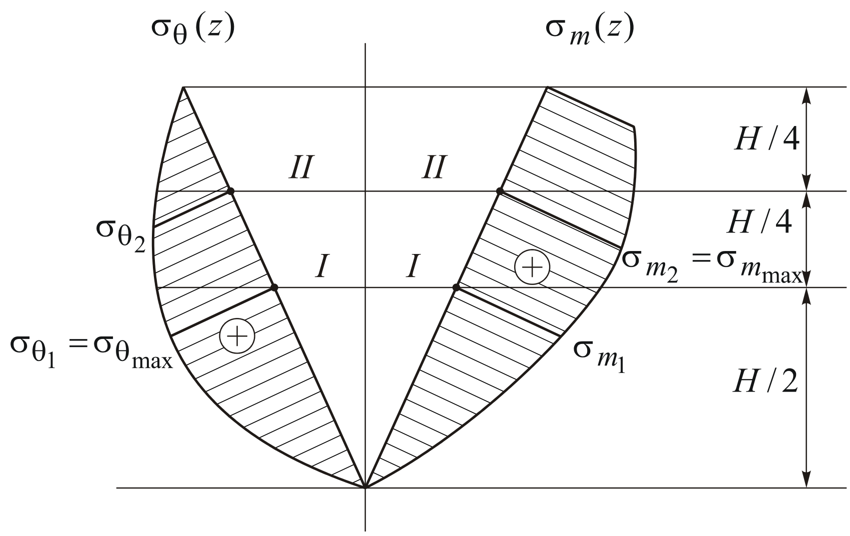

Fig. 8

$$\sum P_z = 0 = 2\pi R(z)\delta\sigma_m(z)\cos\alpha = (H - z)\gamma\pi R^2(z) + \frac{1}{3}\pi R^2(z)z\gamma\,,$$

$$2\delta\sigma_m(z)\cos\alpha = H\gamma R(z) - z\gamma R(z) + \frac{1}{3}R(z)z\gamma = H\gamma R(z) - \frac{2}{3}z\gamma R(z) =$$

$$= \gamma tg\alpha\left(Hz - \frac{2}{3}z^2\right),$$

$$\sigma_m(z) = \frac{\gamma tg\alpha}{2\delta\cos\alpha}\left[Hz - \frac{2}{3}z^2\right],$$

To calculate this stress maximum value, let us determine its derivative and equate to zero:

$$\frac{d\sigma_m}{dz} = \frac{\gamma tg\alpha}{2\delta\cos\alpha}\left(H - \frac{4}{3}z_e\right) = 0, \quad z_e = \frac{3}{4}H\,.$$

The maximum stress $\sigma_{m_{\max}}$ occurs in the section of the conical shell at $z = \frac{3}{4}H$. It's maximum value is determined by the formula:

$$\sigma_{m_{\max}} = \frac{3}{16}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2.$$

Graph of meridian stress distribution is shown on Fig. 8 (right).

Therefore, two potentially critical cross sections must be considered to estimate this shell strength:

section I-I $z=\frac{H}{2}$

$$\sigma_{\theta_1}=\sigma_{\theta_{\max}}=\frac{1}{4}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2, \quad \sigma_{m_1}=\frac{1}{6}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2.$$

section II-II $z=\frac{3H}{4}$

$$\sigma_{\theta_2}=\frac{3}{16}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2, \quad \sigma_{m_2}=\sigma_{m_{\max}}=\frac{3}{16}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2.$$

Applying III strength theory, midsection is critical, since

$$\sigma_1=\sigma_{\theta_{\max}}=\frac{1}{4}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2,$$

$$\sigma_2=\sigma_{m_1}=\frac{1}{6}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2, \quad \sigma_3=0$$

$$\sigma_{eq}^{III}=\frac{1}{4}\frac{\gamma tg\alpha}{\delta\cos\alpha}H^2.$$

Example 4 Compare the strength two cylindrical pressure vessels of the same dimensions under hydraulic pressure with specific weight γ but with different support conditions.

1. **Pressure vessel 1**. It is shown on Fig. 9. Evidently, that meridian and circumferential stresses will be the functions of z coordinate due to hydraulic pressure dependence on the depths of liquid. Stress element is shown on Fig. 10. $\sigma_\theta(z)$ will be found applying Laplace formula:

$$\frac{\sigma_m(z)}{\infty}+\frac{\sigma_\theta(z)}{R}=\frac{p(z)}{\delta}\rightarrow \qquad \sigma_\theta(z)=\frac{p(z)R}{\delta}=\frac{\gamma zR}{\delta},$$

$$\sigma_\theta(0)=0, \quad \sigma_\theta(H)=\frac{\gamma HR}{\delta}.$$

This linear function is shown on Fig. 11 (left).

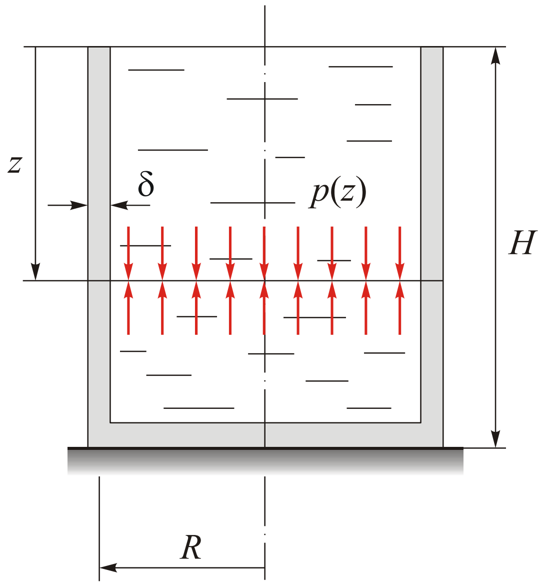

Fig. 9

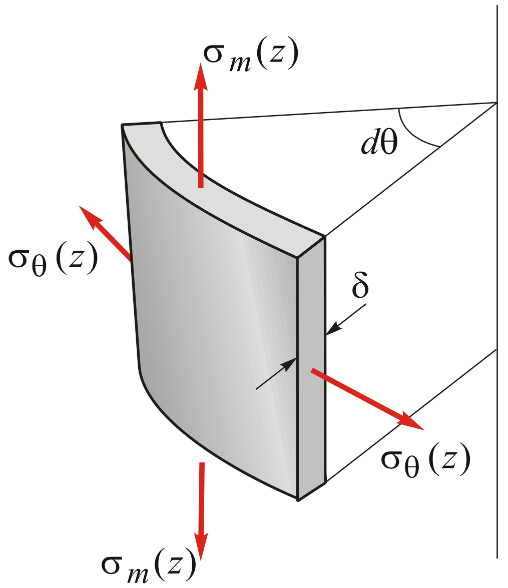

Fig. 10

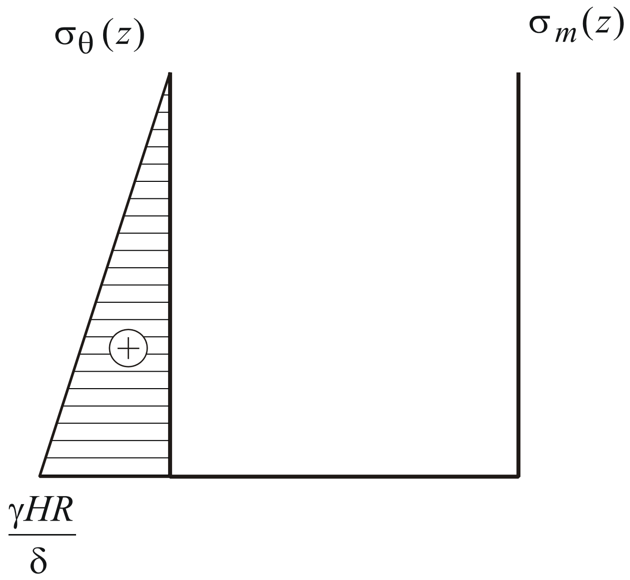

Fig. 11

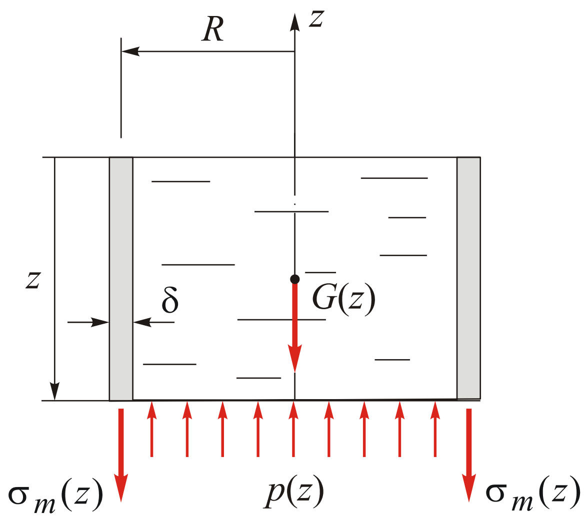

Fig. 12

Meridian stress $\sigma_m(z)$ we will calculate, applying equation of equilibrium for the low part of the vessel shown on Fig. 12:

$$\sum F_z = 0 = p(z)\pi R^2 - \gamma z \pi R^2 - \sigma_m(z) 2\pi R\delta =$$
$$= \gamma z \pi R^2 - \gamma z \pi R^2 - \sigma_m(z) 2\pi R\delta \;\rightarrow\; \sigma_m(z) = 0 .$$

In result, meridian stresses are zero for this type of boundary conditions, and bottom section is critical.

Due to uniaxial stress state in an arbitrary point of the shell we will write following condition of strength:

$$\sigma_{\theta_{\max}} = \frac{\gamma HR}{\delta} \le [\sigma].$$

2. **Pressure vessel 2**. It is shown on Fig. 13.

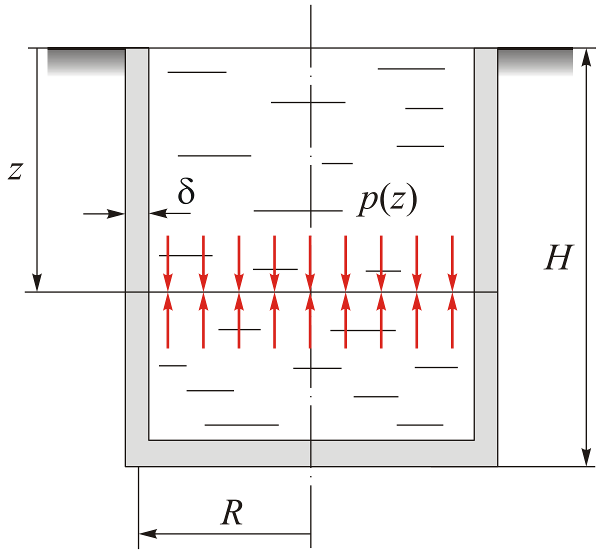

Fig. 13

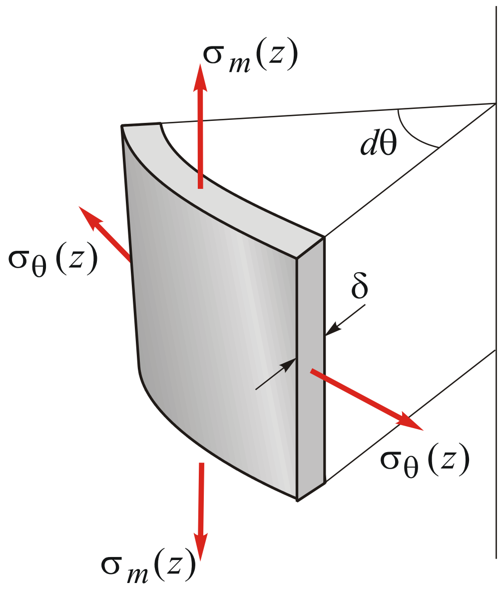

Fig. 14

In this case, Laplace formula will lead to the same formula for $\sigma_\theta(z)$:

$$\sigma_\theta(z) = \frac{p(z)R}{\delta} = \frac{\gamma zR}{\delta},$$

$$\sigma_\theta(0) = 0, \quad \sigma_\theta(H) = \frac{\gamma HR}{\delta}.$$

This linear function is shown on Fig. 15 (left).

Meridional stress $\sigma_m(z)$ we will calculate, applying equation of equilibrium for upper part of the vessel show on Fig. 16:

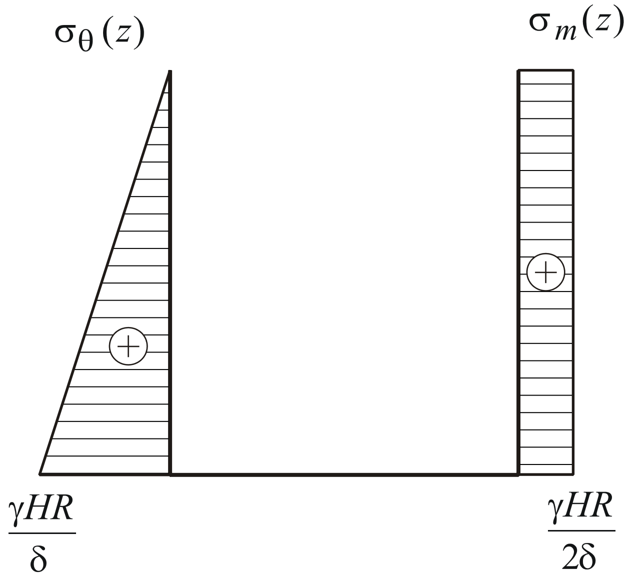

Fig. 15

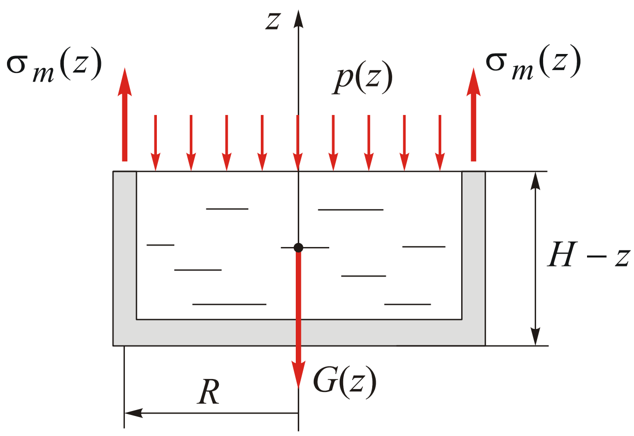

Fig. 16

$$\sum F(z) = 0 = -p(z)\pi R^2 - \gamma(H-z)\pi R^2 + \sigma_m(z)2\pi R\delta =$$

$$= -\gamma z\pi R^2 + \gamma z\pi R^2 - \gamma H\pi R^2 + \sigma_m(z)2\pi R\delta\,.$$

$$\sigma_m(z) = \frac{\gamma HR}{2\delta},$$

$$\sigma_m(0) = \sigma_m(H) = \frac{\gamma HR}{2\delta}.$$

Due to plane stress state in an arbitrary point of the bottom section, we will apply maximum shear stress theory to write following condition of strength:

$$\sigma_1 = \sigma_{\theta_{\max}} = \frac{\gamma HR}{\delta},$$

$$\sigma_2 = \sigma_m = \frac{\gamma HR}{2\delta},$$

$$\sigma_3 = 0\,.$$

$$\sigma_{eq}^{III} = \left(\frac{\gamma HR}{\delta} - 0\right) \le [\sigma].$$

General conclusion. Both shells are equicritical (equidangerous).