MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National aerospace university “Kharkiv Aviation Institute”

Department of aircraft strength

Course
Mechanics of materials and structures

HOME PROBLEM 15

Generalized Displacements in Plane Frames in Plane Bending

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National aerospace university
"Kharkiv Aviation Institute"
Department of aircraft strength

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Full name of the student, group

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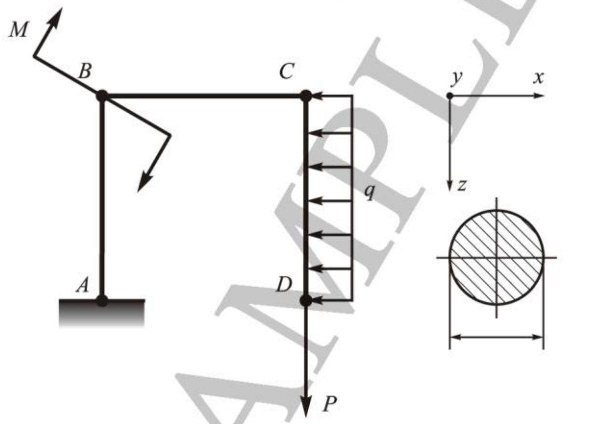

Given: $q = 10$ kN/m; $P = 20$ kN; $M = 10$ kNm; $E = 2\times10^{11}$ Pa; $[\sigma] = 160$ MPa; $a = 2$ m.

Goal: 1) calculate dimensions of the cross-section choosing the one of following: a) diameter of the round solid; b) dimensions of the rectangle (*h*/*b*=2); 2) calculate vertical and horizontal displacements and the slopes in the following points:

θ_B – ?	θ_C – ?	θ_D – ?
z_B – ?	z_C – ?	z_D – ?
x_B – ?	x_C – ?	x_D – ?

signature

Full name of the lecturer

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Given: $q = 10$ kN/m, $F_1 = 20$ kN, $F_2 = 10$ kN, M=30 kNm, $a = 2$ m, $E = 2 \times 10^{11}$ Pa, $[\sigma] = 160$ MPa.

It is necessary:
1) To determine the dimensions of rectangular cross section for the plane frame.
2) To calculate the vertical linear displacement and angle of rotation for two specified cross-sections, i.e. θ_B, z_D.

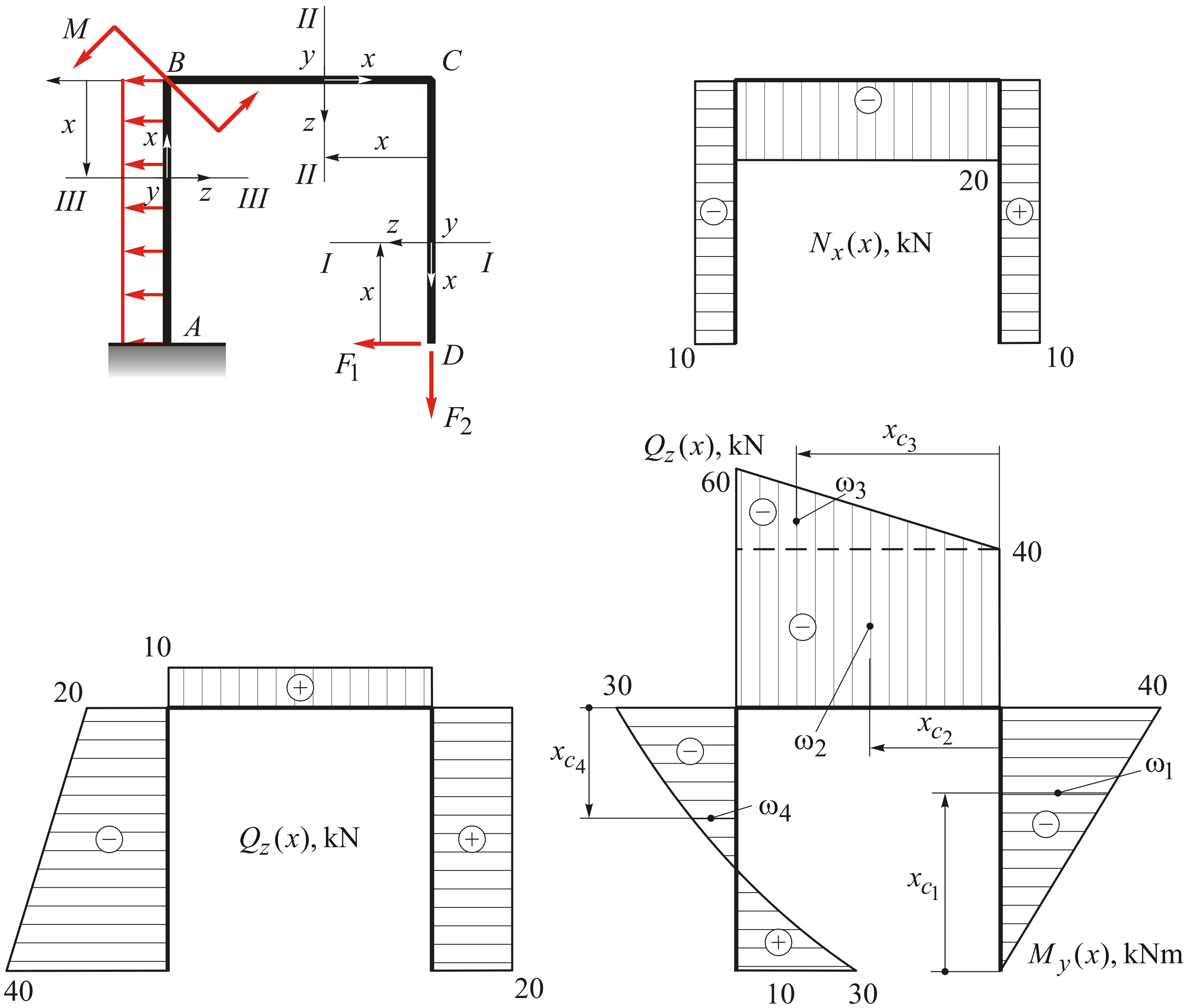


Fig. 1

Note 1

In a plane bending, the Mohr's integral is:

$$\delta = \sum_{i=1}^{n} \int_{l_i} \frac{M_{yF}(x)\bar{M}_y(x)}{EI_y} dx,$$

where $M_{yF}(x)$ – bending moment in an arbitrary section of the force system (F), $\bar{M}_y(x)$ – bending moment in an arbitrary section of the unit system (1);
EI_y – flexural rigidity.

Vereshchagin's formula is:

$\delta = \sum_{i=1}^{n} \frac{\omega_i \eta_i}{EI_y}$, where ω_i – area of the bending moment graph part for the force system; η_i – ordinate of unit diagram under centroid of force system bending moment graph.

Note 2

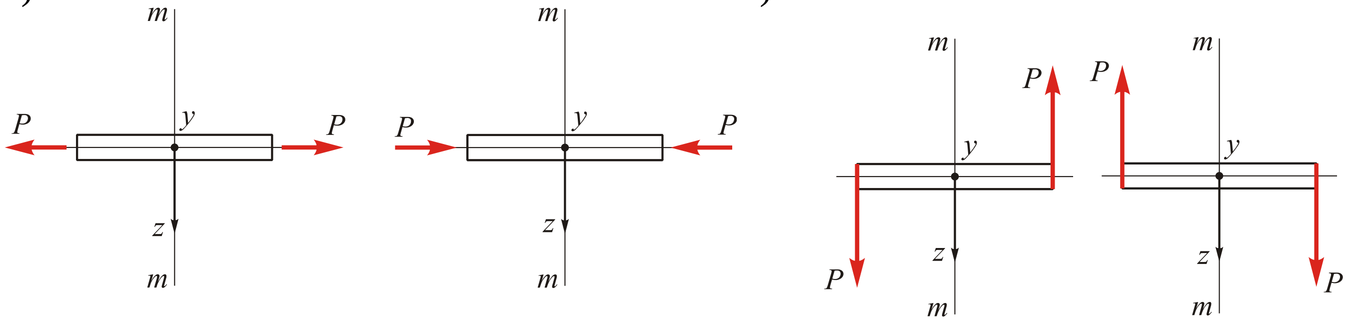
Displacements due to normal force will be omitted in our solution due to their negligibly little effect.

Solution

1. Calculating the rectangular cross-section dimensions from the condition of strength in critical section.

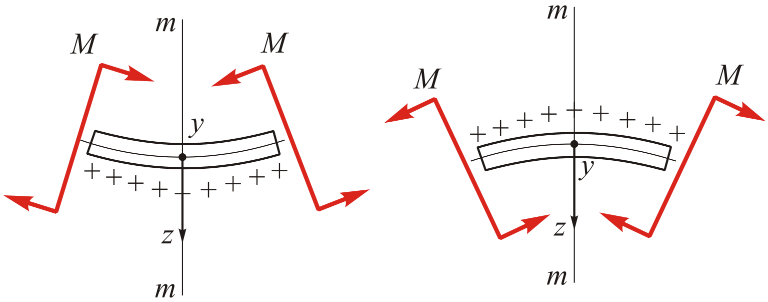
For this purpose we should write the equations of internal forces in an arbitrary sections of specified plane frame which will be considered in future as the force system (F). Also, we will use the following sign conventions (see. Fig. 2):

a) for normal forces b) for shear forces



$N_x^{m-m} > 0$ $N_x^{m-m} < 0$ $Q_z^{m-m} < 0$ $Q_z^{m-m} > 0$

c) for bending moments



$M_y^{m-m} > 0$ $M_y^{m-m} < 0$

Fig. 2

If the curvature of deflected beam is directed alone selected z axis, such bending moment will be considered as positive, and vice versa. Bending moment diagrams will be designed on the stretched fibers (see Fig. 2).

Equations of internal forces are

I - I $0 < x < a$

$N_x^I(x) = +F_2 = +10$ кN,

$Q_z^I(x) = +F_1 = +20$ кN,

$M_y^I(x) = -F_1 x|_{x=0} = 0|_{x=2} = -40$ kNm.

II - II $0 < x < a$

$N_x^{II}(x) = -F_1 = -20$ kN,

$Q_z^{II}(x) = +F_2 = +10$ kN,

$M_y^{II}(x) = -F_1 a - F_2 x|_{x=0} = -40|_{x=2} = -60$ kNm.

III - III $0 < x < a$

$N_x^{III}(x) = -F_2 = -10$ kN,

$Q_z^{III}(x) = -qx - F_1\big|_{x=0} = -20\big|_{x=2} = -40$ kN,

$M_y^{III}(x) = M + q\dfrac{x^2}{2} - F_1(a-x) - F_2 a\big|_{x=0} = -30\big|_{x=2} = +30$ kNm.

To determine the critical section, i.e. section with the largest bending moment, we should design diagrams of internal forces distribution along the length of the frame. This diagrams are shown on Fig. 1. We estimate the accuracy of internal forces calculating, checking the balance of infinite angular connections at points *B* and *C* (see Fig. 3).

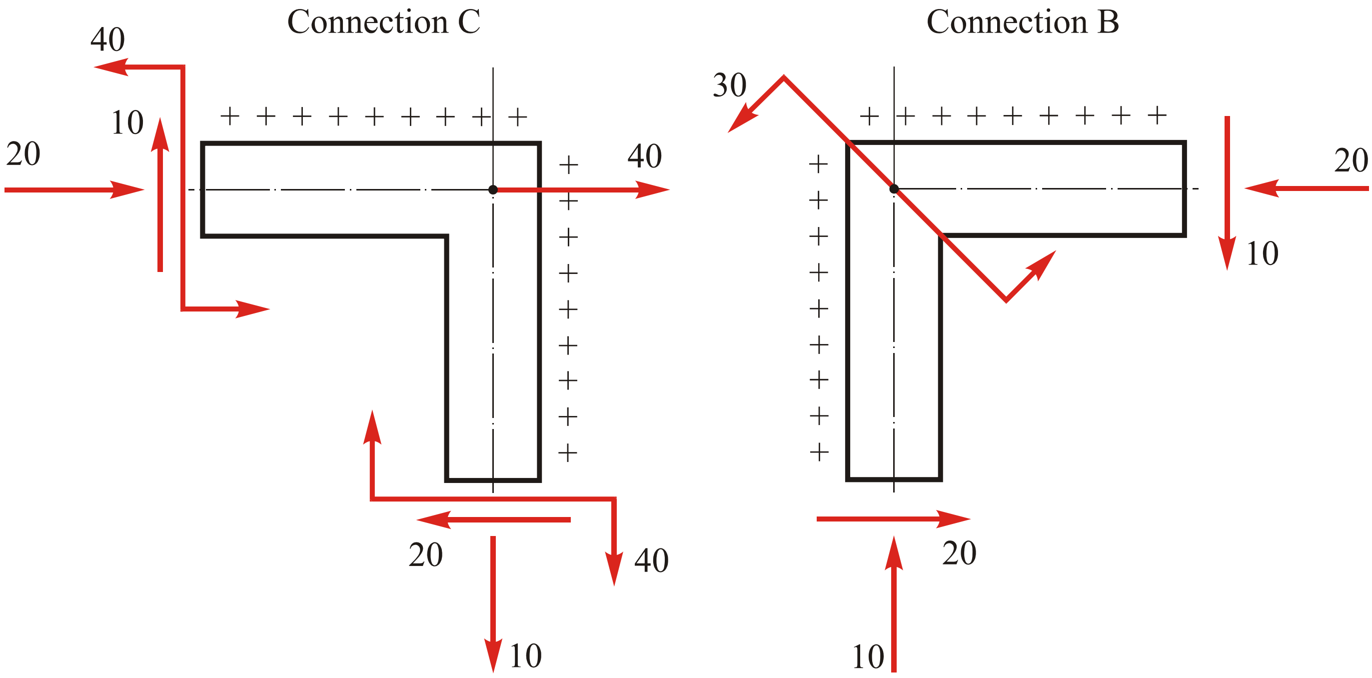


Fig. 3

It follows from the diagram $M_y(x)$ that *B*-section is critical and $|M_{y\max}| = 60$ kNm.

From the condition of strength the dimensions of rectangular cross-section are the following:

$$\sigma_{\max} = \frac{|M_{y\max}|}{W_y} \le [\sigma], \quad W_y = \frac{I_y}{z_{\max}} = \frac{\dfrac{bh^3}{12}}{\dfrac{h}{2}} = \frac{bh^2}{6} = \frac{2b^3}{3} \text{ m}^3, \text{ assuming } \frac{h}{b} = 2.$$

In result, $b = \sqrt[3]{\dfrac{3M_{y\max}}{2[\sigma]}} = \sqrt[3]{\dfrac{3\times 60\times 10^3}{2\times 160\times 10^6}} = 0{,}083$ m, $\quad h = 2\times b = 0{,}166$ м.

For further calculations we should determine the axial moment of inertia of the section selected

$$I_y = \frac{bh^3}{12} = \frac{0{,}083\times 0{,}166^3}{12} = 3160\times 10^{-8} \text{ м}^4.$$

2) Calculating the angle of rotation of *B*-section. In this calculations, specified frame will be considered as the force system (F), and corresponding unit system (1) will be designed by applying unit dimensionless moment $\overline{M}=1$ in *B*-section. Unit system (1) is shown on the Fig. 4. Let us write one more the equations of bending moments for force system in the most simple shape and also the equations of bending moments for the unit system. Note, that these equations must be written in identical systems of coordinates for further substituting into Mohr's integral. Bending moment diagram for the force system is shown on Fig. 1, and for unit system – on Fig. 4.

I - I $\quad 0<x<a$

$M_{yF}^{I}(x)=-F_1x=-20x,$ kNm,

$\overline{M_x^{I}}(x)=0.$

II - II $\quad 0<x<a$

$M_{yF}^{II}(x)=-F_1a-F_2x=-40-10x,$ kNm,

$\overline{M_y^{II}}(x)=0.$

III - III $\quad 0<x<a$

$M_{yF}^{III}(x)=M+q\frac{x^2}{2}-F_1(a-x)-F_2a=5x^2+20x-30$ kNm,

$\overline{M_y^{III}}(x)=1.$

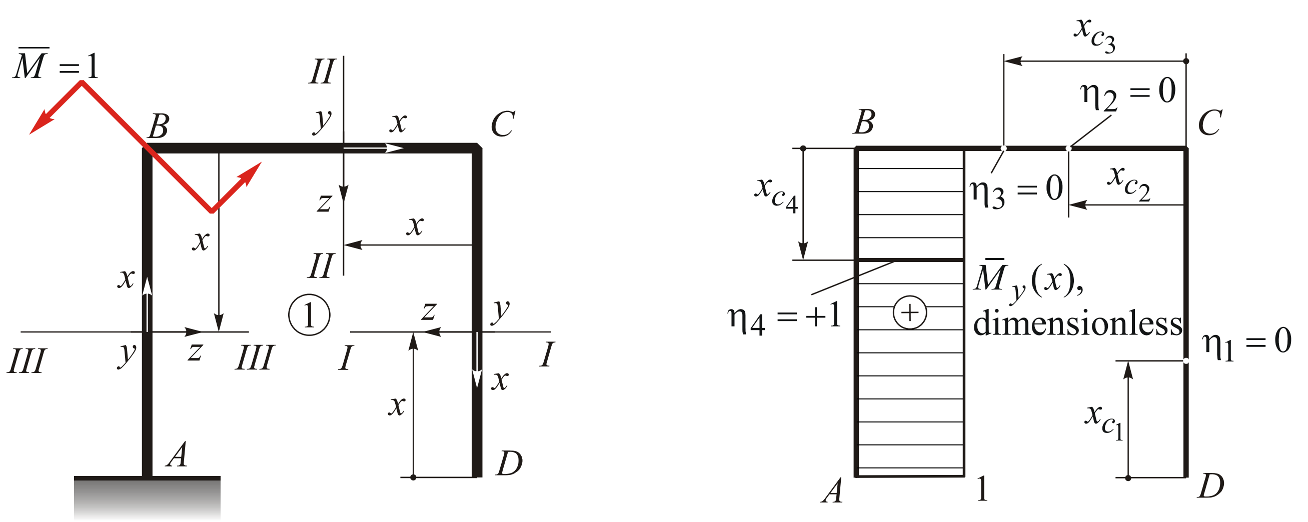


Fig. 4

Let us substitute the equations of bending moments for the force and unit systems in Mohr’s integral to determine angle of *B*-section rotation:

$$\theta_B=\frac{1}{EI_y}\left[\int_0^a M_{yF}^{I}(x)\overline{M_y^{I}}dx+\int_0^a M_{yF}^{II}(x)\overline{M_y^{II}}dx+\int_0^a M_{yF}^{III}(x)\overline{M_y^{III}}dx\right]=$$

$$=\frac{1}{EI_y}\left[\int_0^a(-20x)(0)dx+\int_0^a(-40-10x)(0)dx+\int_0^a\left(5x^2+20x-30\right)(1)dx\right]=$$

$$=\frac{1}{EI_y}\left[5\int_0^a x^2dx+20\int_0^a xdx-30\int_0^a dx\right]=\frac{1}{EI_y}\left[5\frac{x^3}{3}\Big|_0^2+20\frac{x^2}{2}\Big|_0^2-30x\Big|_0^2\right]=$$

$$=\frac{1}{EI_y}\left[5\times\frac{2^3}{3}+20\times\frac{2^2}{2}-30\times 2\right]=-\frac{20}{3EI_y},\ \frac{\text{kNm}^2}{EI_y}.$$

After substituting the parameters of flexural rigidity, we obtain

$$\theta_B=-\frac{20\times 10^3}{3\times 2\times 10^{11}\times 3160\times 10^{-8}}=1.05\times 10^{-3}\ \text{rad.}$$

Note, that the result minus sign means that it is really opposite to the direction of the $\bar{M}=1$ moment applied to the unit system, i.e. in reality, the *B* cross-section is rotated clockwise.

To calculate this angle of rotation by Vereshchagin's method, first construct the bending moment diagrams for the force and unit systems, using $M_{y_{Fi}}(x)$ and $\overline{M_{y_i}}(x)$ equations. For the force system, the diagram is shown on Fig. 1 and for the unit system – on Fig. 4. Unit bending moment diagram will be used to determine it's four ordinates η_1, η_2, η_3, η_4, under the centroids of the force system diagram parts ω_1, ω_2, ω_3, ω_4, i.e. in the points with coordinates x_{c_1}, x_{c_2}, x_{c_3}, x_{c_4} (see Figs. 1 and 4). In order to simplify the solution, trapezoidal bending moment diagram $M_{yF}^{II}(x)$ will be divided into two simple shapes – triangle and rectangle with known positions of their centroids of gravity.

Before substituting in Vereshchagin's formula, we should calculate the areas of bending moment diagrams for the force system, i.e. ω_1, ω_2, ω_3, ω_4 values:

$$\omega_1=\frac{(-40)\times 2}{2}=-40\ \text{kNm}^2,$$

$$\omega_2=(-40)\times 2=-80\ \text{kNm}^2,$$

$$\omega_3=\frac{(-20)\times 2}{2}=-20\ \text{kNm}^2.$$

Since diagram $M_{yF}^{IV}(x)$ is limited by curved contour, to find the area ω_4 we need to apply integration. Note, that equation $M_{yF}^{IV}(x)$ is written in the coordinate system with *B* point origin. Then

$$\omega_4=\int_0^a M_{yF}^{IV}(x)dx=\int_0^2(5x^2+20x-30)dx=5\int_0^2 x^2dx+20\int_0^2 xdx-30\int_0^2 dx=$$

$$=5\frac{x^3}{3}\Big|_0^2+20\frac{x^2}{2}\Big|_0^2-30x\Big|_0^2=5\times\frac{8}{3}+20\times 2-30\times 2=-\frac{20}{3},\ \text{kNm}^2.$$

To simplify further calculations, let us represent the equation $M_{yF}^{IV}(x)$ with corresponding area ω_4 in a manner, shown on the Fig. 5, i.e. consisting of two parts ω_4' и ω_4''.

Obviously, that $\omega_4''=(-30)\times 2=-60\ \text{kNm}^2$ and $x_{c_4}''=1$ m.

To find ω_4' we should use the integration:

$$\omega_4' = \int_0^2 (5x^2 + 20x)dx = 5\int_0^2 x^2 dx + 20\int_0^2 x dx = 5\frac{x^3}{3}\Big|_0^2 + 20\frac{x^2}{2}\Big|_0^2 = 5\times\frac{2^3}{3} + 20\times\frac{2^2}{2} = \frac{160}{3}, \text{kNm}^2.$$

There is no need in finding the coordinates x_{c_4}' and x_{c_4}'', because diagram $\overline{M}_y^{III}(x)$ is constant along the length of *III-III* portion (see Fig. 4), i.e. $\eta_4' = +1$ и $\eta_4'' = +1$. Then finally the Vereshchagin's formula becomes:

$$\theta_B = \frac{1}{EI_y}\left[\omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3 + \omega_4'\eta_4' + \omega_4''\eta_4''\right] =$$

$$= \frac{1}{EI_y}\left[(-40)(0) + (-80)(0) + (-20)(0) + \left(+\frac{160}{3}\right)(+1) + (-60)(+1)\right] = -\frac{20}{3EI_y},\ \frac{\text{kNm}^2}{EI_y},$$

which corresponds to the result previously obtained by Mohr's method.

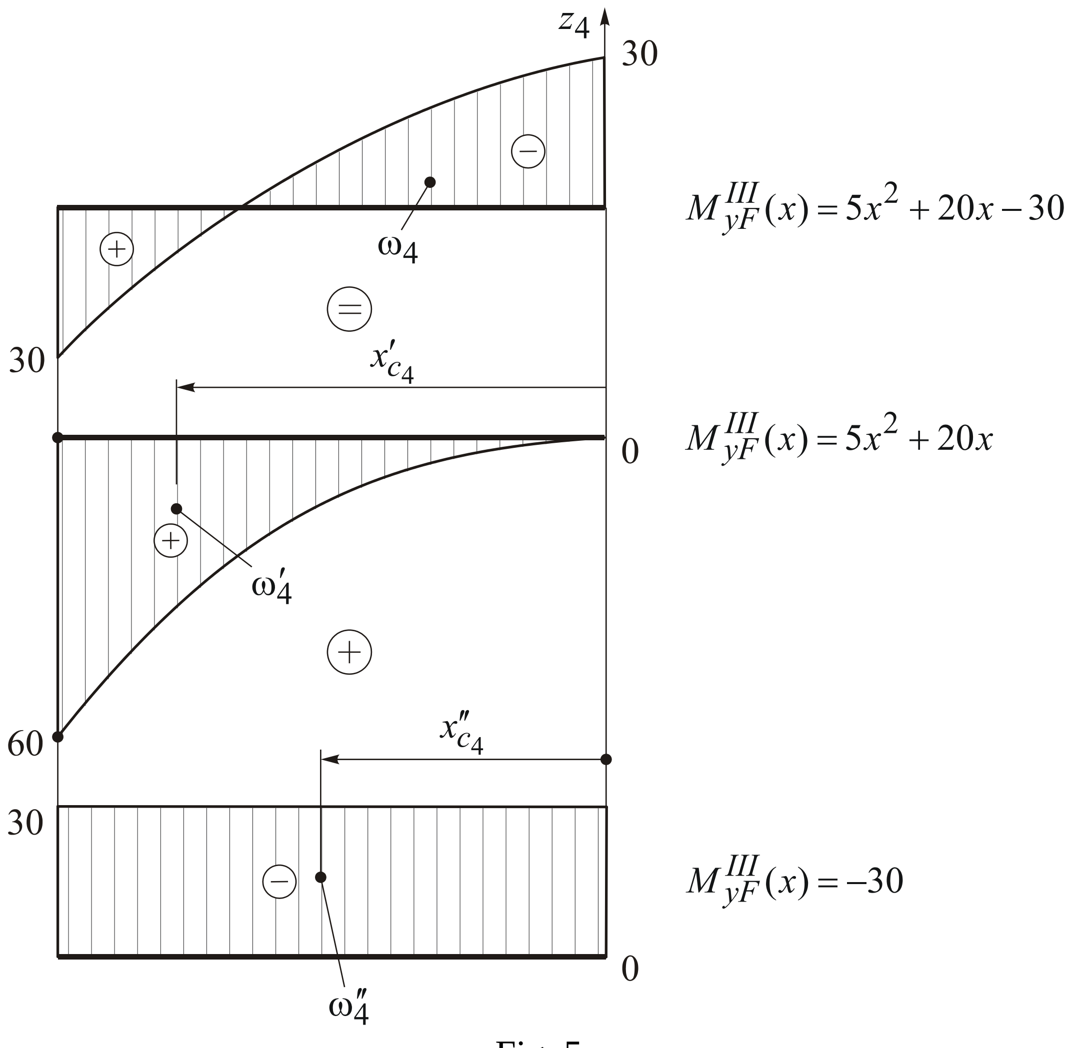


Fig. 5

3) Calculating the vertical component of the *D* point linear displacement, i.e. z_D value. For this purpose, specified frame will also be used as the force system (*F*), but new unit system (1) should be designed by the unit dimensionless force $\overline{F} = 1$ applying in *D* point in vertical direction, for example, downwards (see Fig. 6).

Let us write again the equations of bending moments for the force system $M_{yF_i}(x)$ and the equations of bending moments for the new unit system $\overline{M}_{y_i}(x)$, written in identical systems of coordinates. Equations should be simplified before substitution into the Mohr's integral.

$I-I \quad 0<x<a$

$M_{yF}^{I}(x)=-F_1x=-20x$, kNm,

$\overline{M}_{y}^{I}(x)=0$.

$II-II \quad 0<x<a$

$M_{yF}^{II}(x)=-F_1a-F_2x=-40-10x$, kNm,

$\overline{M}_{y}^{II}(x)=1\times x$, m.

$III-III \quad 0<x<a$

$M_{yF}^{III}(x)=M+q\frac{x^2}{2}-F_1(a-x)-F_2a=5x^2+20x-30$, kNm,

$\overline{M}_{y}^{III}=-1\times a=-2$ m.

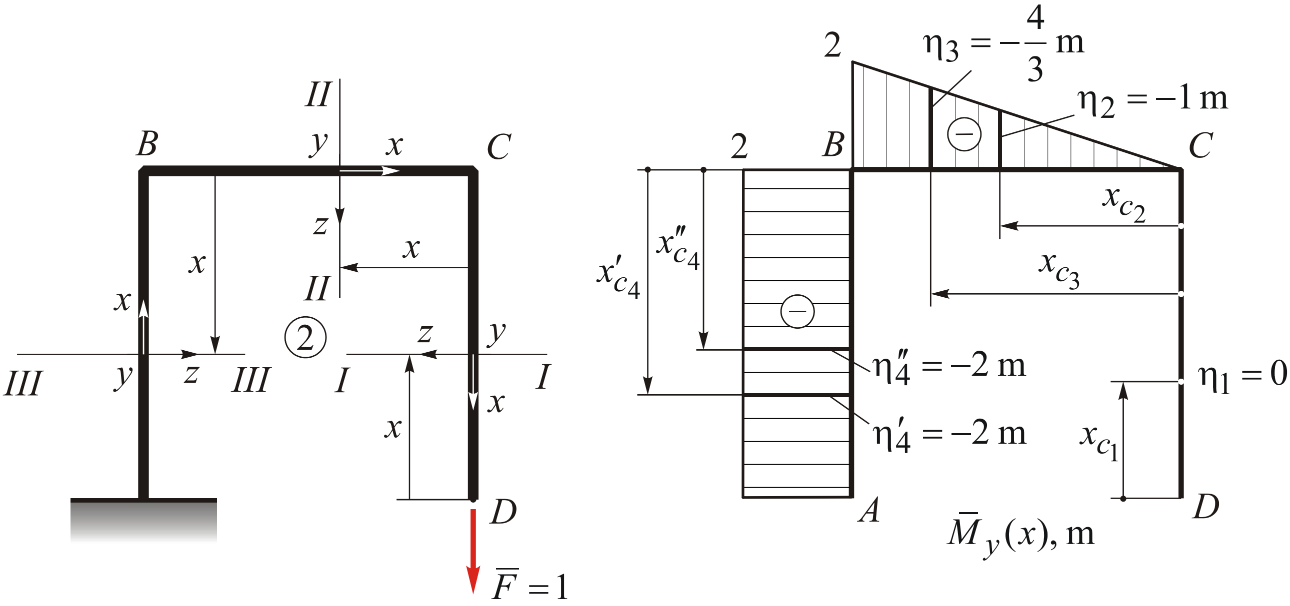


Fig. 6

Mohr's integral is the following:

$$z_D=\frac{1}{EI_y}\left[\int_0^a M_{yF}^{I}(x)\overline{M}_{y}(x)dx+\int_0^a M_{yF}^{II}(x)\overline{M}_{y}^{II}(x)dx+\int_0^a M_{yF}^{III}(x)\overline{M}_{y}^{III}(x)dx\right]=$$

$$=\frac{1}{EI_y}\left[\int_0^a(-20x)(0)dx+\int_0^a(-40-10x)(-x)dx+\int_0^a\left(5x^2+20x-30\right)(-2)dx\right]=$$

$$\frac{1}{EI_y}\left[40\int_0^a xdx+10\int_0^a x^2dx-10\int_0^2 x^2dx-10\int_0^2 xdx+60\int_0^2 dx\right]=$$

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$$= \frac{1}{EI_y}\left[60x\,|_0^2\right] = \frac{1}{EI_y}\left[60\times 2\right] = +\frac{120}{EI},\ \frac{\text{kNm}^3}{EI_\text{y}}.$$

After substituting the parameters of flexural rigidity the result is

$$z_D = +\frac{120\times 10^3}{2\times 10^{11}\times 3160\times 10^{-8}} = +0.019\ \text{m} = +19\ \text{mm}.$$

Note, that "+" sign of the result means that the actual direction of point *D* displacement coincides with the direction of $\overline{F}=1$ force, i.e. point *D* moves downwards.

To calculate this displacement by Vereshchagin's method, we should design the diagram of unit bending moments $\overline{M}(x)$, using equations $\overline{M}_{y_i}(x)$. It is shown on Fig. 6 (right). We will use it to determine the five ordinates $\eta_1, \eta_2, \eta_3, \eta_4', \eta_4''$ which are situated under the centroids of the force system (F) diagram parts, i.e. in the points with coordinates $x_{c_1}, x_{c_2}, x_{c_3}, x_{c_4}', x_{c_4}''$. To simplify the solution, trapezoidal bending moment diagram $M_{yF}^{II}(x)$ should be divided on a triangle and a rectangle with known positions of their centroids $x_{c_2} = 1$ m and $x_{c_3} = 4/3$ m. Function $M_{yF}^{III}(x)$ will again be presented, as shown on Fig. 5. Coordinates of the centroids for the areas ω_4' and ω_4'', i.e. x_{c_4}' and x_{c_4}'' values not need to be defined, since unit diagram $\overline{M}_y^{III}(x)$ in this case also is constant along the length of portion *III-III* (see Fig. 6), i.e. $\eta_4' = -2$ m, $\eta_4'' = -2$ m. At the same time, $\eta_1 = 0$ m, $\eta_2 = -1$ m, $\eta_3 = -4/3$ m (see Fig. 6).

As a result, the Vereshchagin's formula becomes:

$$z_D = \frac{1}{EI_y}\left[\omega_1\eta_1 + \omega_2\eta_2 + \omega_3\eta_3 + \omega_4'\eta_4' + \omega_4''\eta_4''\right] =$$

$$= \frac{1}{EI_y}\left[(-40)(0) + (-80)(-1) + (-20)\left(-\frac{4}{3}\right) + \left(+\frac{160}{3}\right)(-2) + (-60)(-2)\right] = +\frac{120}{EI_y},\ \frac{\text{kNm}^3}{EI_y}.$$

This result corresponds to previously performed calculations by Mohr's method.