MINISTRY OF EDUCATION AND SCIENCE OF UKRAINE

National aerospace university “Kharkiv Aviation Institute”
Department of aircraft strength

Course
Mechanics of materials and structures

HOME PROBLEM 16

Internal Forces in Statically Indeterminate Plane Frame

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National aerospace university
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Department of aircraft strength

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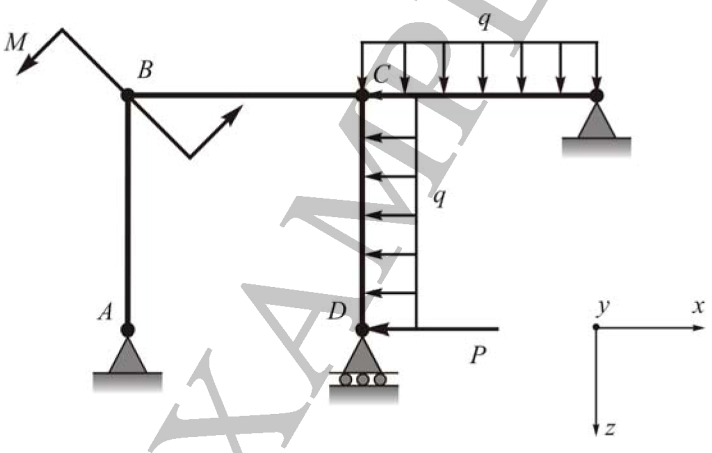


Given: $q = 10$ kN/m; $P = 20$ kN; $M = 10$ kNm; $l = 2$ m.
Goal: 1) open static indeterminacy using the force method and draw the graphs $N_x(x), Q_z(x), M_y(x)$.

signature
Full name of the lecturer

Mark:

Solution

(1) Degree of static indeterminacy

$k = m - n$, where $m = 5$ – total number of constraints,

$n = 3$ – minimum number of constraints.

After substituting $k = 5 - 3 = 2$.

Conclusion: plane frame is 2-fold statically indeterminate.

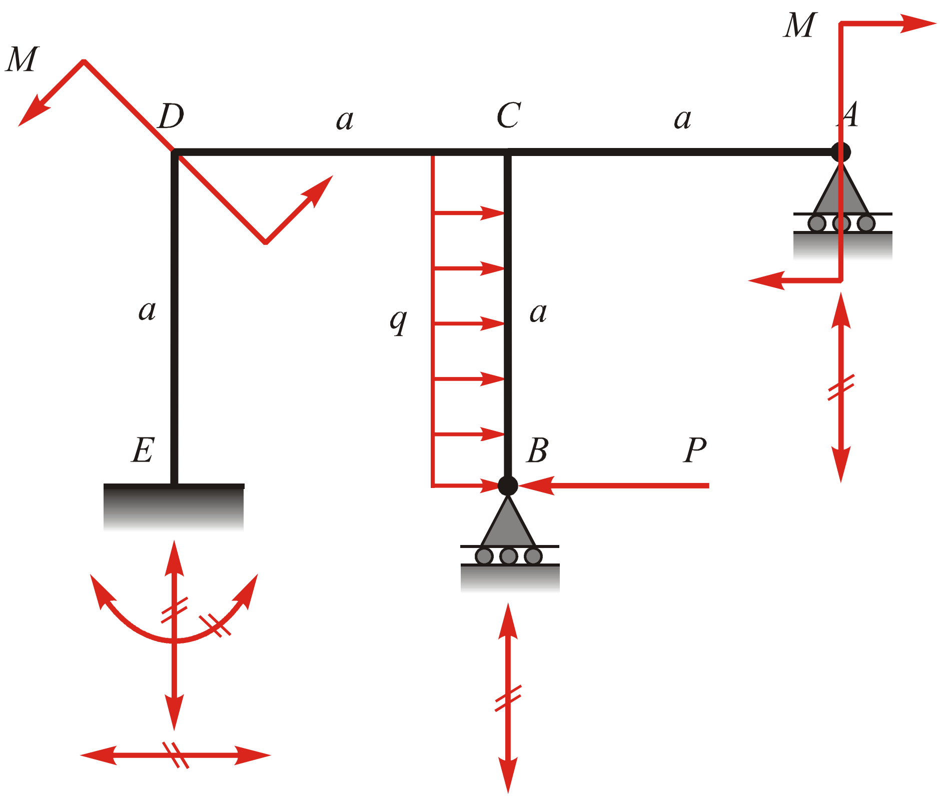


Fig. 1 Plane frame in equilibrium under external loading and reactions of supports

(2) Selecting the one of base systems

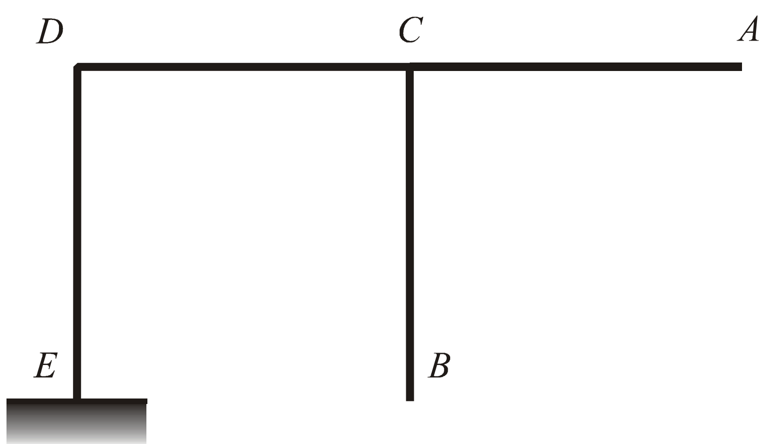


Fig. 2 Selected base system

Note. Base system should be statically determinate.

(3) Designing the equivalent system

Note. To design the equivalent system, it is necessary to impose on the base system external forces and also the reactions of redundant constrains X_1 and X_2.

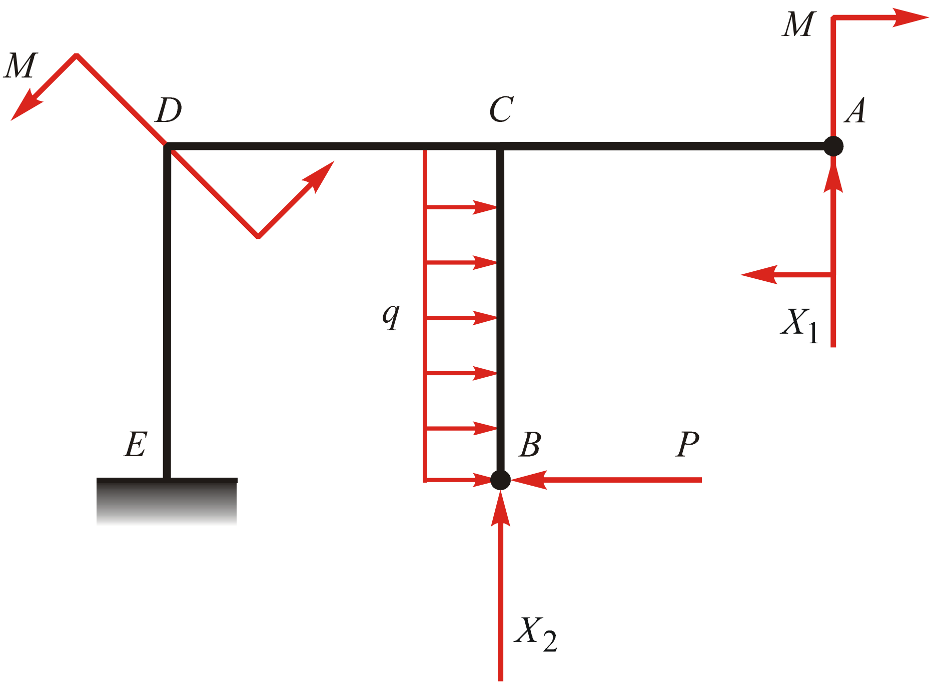


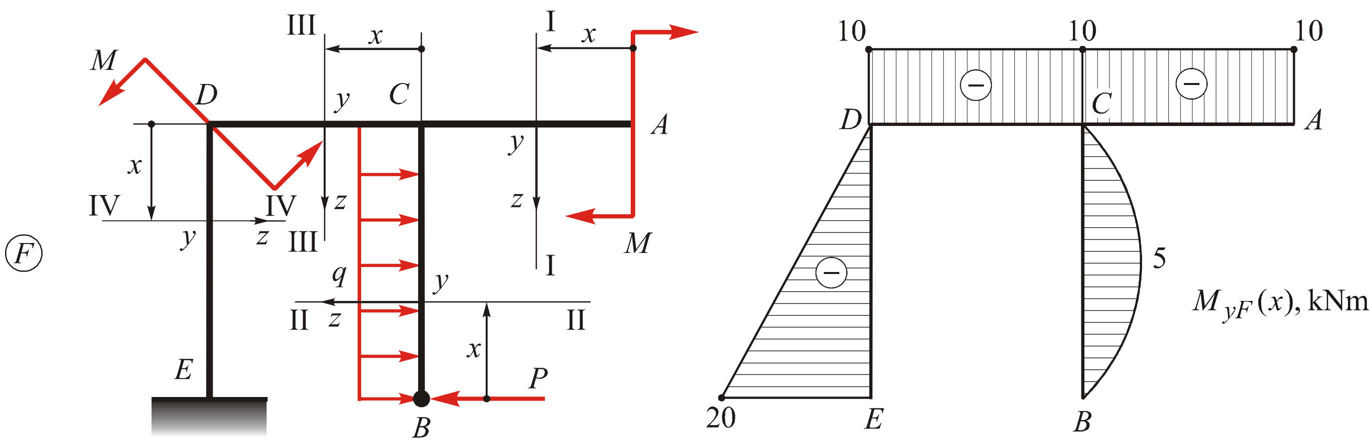
Fig. 3 Designed equivalent system

(4) Writing the system of canonical equations (compatibility equations) taking into consideration evidently zero vertical displacements of *A* and *B* points in given system.

$$\begin{cases} d_{vert \cdot A}(X_1, X_2, F) = 0, \\ d_{vert \cdot B}(X_1, X_2, F) = 0, \end{cases} \quad \text{or in canonical shape} \quad \begin{cases} d_{11}X_1 + d_{12}X_2 + D_{1F} = 0, \\ d_{21}X_1 + d_{22}X_2 + D_{2F} = 0. \end{cases}$$

(5) Calculating the coefficients of canonical equations.

To find six coefficients d_{11}, d_{12}, d_{21}, d_{22}, D_{1F}, D_{2F}, it is necessary to consider the force system (*F*) and two unit systems: (1) and (2). These systems are shown on Fig 4.



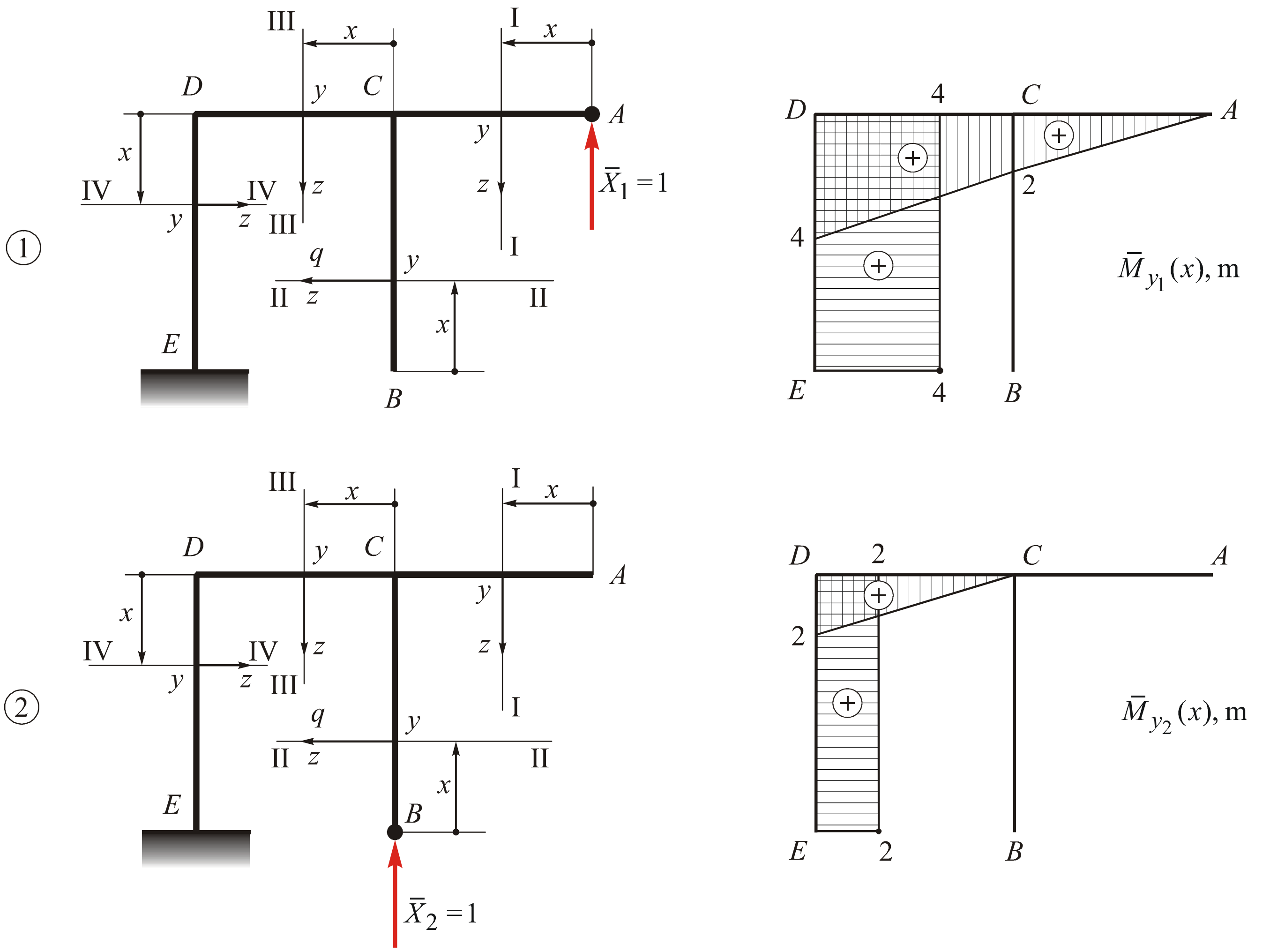


Fig. 4 The force system (F) (above left) and two unit systems with corresponding graphs of bending moments

By applying the method of sections the equations of internal forces are

Portion I – I $(0 < x < 2 \text{ m})$

$$M^{I}_{yF}(x) = -M = -10 \text{ kNm},$$

$$\overline{M}^{I}_{y_1}(x) = +\overline{X}_1 x = 1x = x\big|_{x=0} = 0\big|_{x=2} = 2 \text{ m (linear function)},$$

$$\overline{M}^{I}_{y_2}(x) = 0.$$

Portion II – II $(0 < x < 2 \text{ m})$

$$M^{II}_{yF}(x) = -Px + qx^2/2 = -10x + 5x^2 = 5x^2 - 10x\big|_{x=0} = 0\big|_{x=2} = 2 \text{ kNm (parabola)},$$

$$\overline{M}^{II}_{y_1}(x) = 0,$$

$$\overline{M}^{II}_{y_2}(x) = 0.$$

Portion III – III $(0 < x < 2 \text{ m})$

$$M^{III}_{yF}(x) = -M - Pa + qa^2/2 = -10 - 20 + 20 = -10 \text{ kNm},$$

$$\overline{M}^{III}_{y_1}(x) = +\overline{X}_1(a + x) = 1(2 + x) = 2 + x\big|_{x=0} = 2\big|_{x=2} = 4 \text{ m (linear function)},$$

$$\overline{M}^{III}_{y_2}(x) = +\overline{X}_2 x = x\big|_{x=0} = 0\big|_{x=2} = 2 \text{ m (linear function)}.$$

Portion IV – IV $(0 < x < 2\ \text{m})$

$$M_{yF}^{IV}(x) = +M - M - P(a-x) + qa(a/2 - x) = -10(2-x) + 20(1-x) =$$

$$= -20 + 10x + 20 - 20x = -10x\big|_{x=0} = 0\big|_{x=2} = -20\ \text{kNm (linear function)},$$

$$\overline{M}_{y_1}^{IV}(x) = +\overline{X}_1 2a = 4\ \text{m},$$

$$\overline{M}_{y_2}^{IV}(x) = +\overline{X}_2 a = 2\ \text{m}.$$

To simplify further solution, rewrite the equations inside the Table.

Table

Number of the portion:	Length, m	$M_{yF}(x)$, kNm	$\overline{M}_{y_1}(x)$, m	$\overline{M}_{y_2}(x)$, m
I-I	$0 < x < 2$	-10	x	0
II-II	$0 < x < 2$	$5(x^2 - 2x) = = 5x^2 - 10x$	0	0
III-III	$0 < x < 2$	-10	$2 + x$	x
IV-IV	$0 < x < 2$	$-10x$	4	2

(6) Designing the graphs of bending moments for the force and 2 unit systems (see Fig. 4).
(7) Calculating the coefficients of canonical equations using Mohr's method.

$$d_{11} = \frac{1}{EI}\left(\int_0^2 x^2 dx + \int_0^2 (2+x)^2 dx + \int_0^2 16 dx\right) = \frac{1}{EI}\left(\int_0^2 x^2 dx + \int_0^2 (x^2 + 4x + 4) dx + 16\int_0^2 dx\right) =$$

$$= \frac{1}{EI}\left(\frac{x^3}{3} + \frac{x^3}{3} + \frac{4x^2}{2} + 4x + 16x\right)\Bigg|_0^2 = \frac{1}{EI}\left(\frac{8}{3} + \frac{8}{3} + 8 + 8 + 32\right) = \frac{1}{EI}\left(\frac{16}{3} + 48\right) = \frac{160}{3EI}.$$

$$d_{12} = d_{21} = \frac{1}{EI}\left(\int_0^2 (2+x)x dx + \int_0^2 8 dx\right) = \frac{1}{EI}\left(\int_0^2 (2x + x^2) dx + 8\int_0^2 dx\right) =$$

$$= \frac{1}{EI}\left(\frac{2x^2}{2} + \frac{x^3}{3} + 8x\right)\Bigg|_0^2 = \frac{1}{EI}\left(4 + \frac{8}{3} + 16\right) = \frac{1}{EI}\left(20 + \frac{8}{3}\right) = \frac{68}{3EI}.$$

$$d_{22} = \frac{1}{EI}\left(\int_0^2 x^2 dx + \int_0^2 4 dx\right) = \frac{1}{EI}\left(\frac{x^3}{3} + 4x\right)\Bigg|_0^2 = \frac{1}{EI}\left(\frac{8}{3} + 8\right) = \frac{32}{3EI}.$$

$$D_{1F} = \frac{1}{EI}\left(\int_0^2 (-10x) dx + \int_0^2 -10(2+x) dx + \int_0^2 (-40x) dx\right) =$$

$$= \frac{1}{EI}\left(-10\int_0^2 x dx - 10\int_0^2 (2+x) dx - 40\int_0^2 x dx\right) = \frac{1}{EI}\left(-\frac{10x^2}{2} - 10\left(2x + \frac{x^2}{2}\right) - \frac{40x^2}{2}\right)\Bigg|_0^2 =$$

$$= \frac{1}{EI}\left(-5x^2 - 20x - 5x^2 - 20x^2\right)\Big|_0^2 = \frac{1}{EI}(-20 - 40 - 20 - 80) = -\frac{160}{EI}.$$

$$D_{2F}=\frac{1}{EI}\left(\int_0^2(-10x)dx+\int_0^2-20xdx\right)=\frac{1}{EI}\left(-10\int_0^2 xdx-20\int_0^2 xdx\right)=$$

$$=\frac{1}{EI}\left(-\frac{10x^2}{2}-\frac{20x^2}{2}\right)\Bigg|_0^2=\frac{1}{EI}\left(-5x^2-10x^2\right)\Big|_0^2=\frac{1}{EI}(-20-40)=-\frac{60}{EI}.$$

(8) Calculating the coefficients of canonical equations using graphical method.

$$D_{1F}=\frac{1}{EI}\left((-10\times2)\times(+1)+(0)+(-10\times2)\times(+3)+\left(-\frac{20\times2}{2}\right)\times(+4)\right)=-\frac{160}{EI},$$

$$D_{2F}=\frac{1}{EI}\left((0)+(0)+(-10\times2)\times(+1)+\left(-\frac{20\times2}{2}\right)\times(+2)\right)=-\frac{60}{EI},$$

$$d_{11}=\frac{1}{EI}\left(\left(+\frac{4\times4}{2}\right)\times\left(+\frac{2}{3}\times4\right)+(+4\times2)\times(+4)\right)=\frac{160}{3EI},$$

$$d_{22}=\frac{1}{EI}\left(\left(+\frac{2\times2}{2}\right)\times\left(+\frac{2}{3}\times2\right)+(+2\times2)\times(+2)\right)=\frac{32}{3EI},$$

$$d_{12}=\frac{1}{EI}\left(\left(+\frac{2\times2}{2}\right)\times(0)+(+2\times2)\times(+1)+\left(+\frac{2\times2}{2}\right)\times\left(+\frac{2}{3}\times2\right)+(+4\times2)\times(+2)\right)=\frac{68}{3EI}.$$

$$d_{21}=d_{12}=\frac{68}{3EI}.$$

(9) Substituting the coefficients into canonical equations to find X_1 and X_2.

a) First canonical equation is:

$$d_{11}X_1+d_{12}X_2+D_{1F}=0,$$

$$\frac{160}{3EI}X_1+\frac{68}{3EI}X_2-\frac{160}{EI}=0,$$

$$\frac{160}{3}X_1+\frac{68}{3}X_2-160=0,$$

$$160X_1+68X_2-480=0,$$

$$40X_1+17X_2-120=0.$$

$$X_1=\frac{120-17X_2}{40}. \qquad (*)$$

b) Second canonical equation is:

$$d_{21}X_1+d_{22}X_2+D_{2F}=0,$$

$$\frac{68}{3EI}X_1+\frac{32}{3EI}X_2-\frac{60}{EI}=0,$$

$$\frac{68}{3}X_1+\frac{32}{3}X_2-60=0,$$

$$68X_1+32X_2-180=0,$$

$$17X_1+8X_2-45=0.$$

Substituting the value of X_1 from (*) into last equation leads to

$$17\left(\frac{120-17X_2}{40}\right)+8X_2=45,$$

$$\frac{17}{40}(120-17X_2)+8X_2=45,$$

$$\frac{2040}{40}-\frac{289X_2}{40}+8X_2=45,$$

$$2040-289X_2+320X_2=1800,$$

$$31X_2=-240,$$

$$X_2=-\frac{240}{31}=-7.742\ \text{kNm}.$$

After substituting the X_2 value in equation (*) we have

$$X_1=\frac{120-17\left(-\frac{240}{31}\right)}{40}=\frac{120+\frac{4080}{31}}{40}=\frac{\frac{3720+4080}{31}}{40}=\frac{7800}{31\cdot 40}=\frac{7800}{1240}=6.290\ \text{kNm}.$$

As the value of X_2 is negative, its original direction in equivalent system must be changed on opposite.

Conclusion: static indeterminacy is opened.

(10) Calculating the internal forces in statically determinate equivalent system shown on Fig 5.

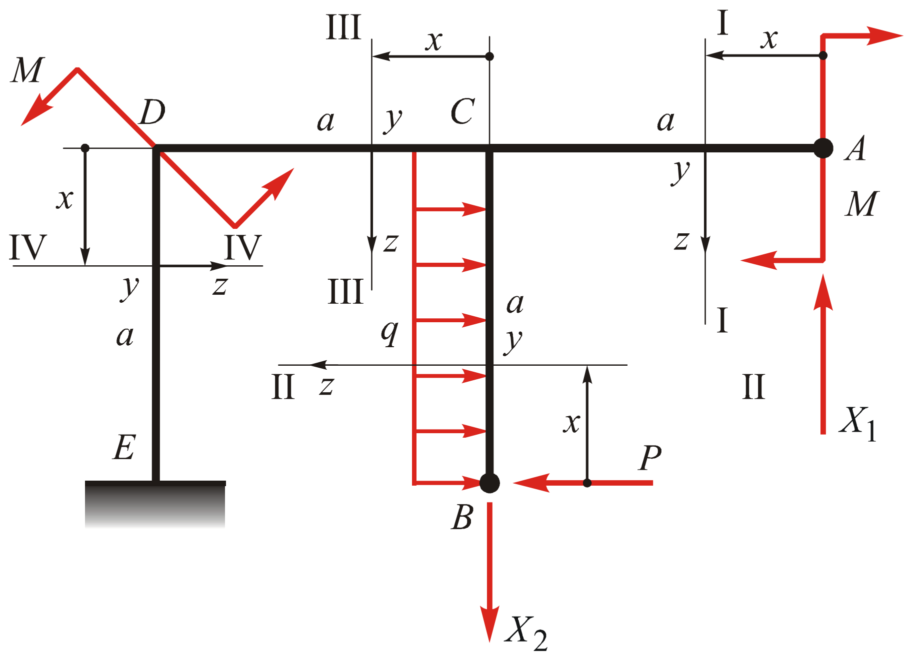


Fig. 5

Portion I – I $(0<x<2\ \text{m})$

$$N_x^I(x)=0\ \text{kN},$$

$$Q_z^I(x)=-X_1=-6.29\ \text{kN},$$

$$M_y^I(x)=+X_1x-M=6.29x-10\big|_{x=0}=-10\big|_{x=2}=2.58\ \text{kNm (linear function).}$$

Portion II – II $(0 < x < 2\ \text{m})$

$N_x^{II}(x) = +X_2 = 7.742$ kN,

$Q_z^{II}(x) = +P - qx = 10 - 10x\big|_{x=0} = 10\big|_{x=2} = -10$ kNm (linear function),

$M_y^{II}(x) = -Px + qx^2/2 = -10x + 5x^2\Big|_{x=0} = 0\big|_{x=2} = -20 + 20 = 0$ kNm (parabola).

Note, that shear force graph intersects the *x* axis. In such case maximum bending moment should be found:

(a) The cross-section of maximal moment is determined by equating to zero the shear force equation:

$Q_z^{II}(x_e) = 0, \quad P - qx_e = 0, \quad qx_e = P, \quad x_e = 1$ m.

(b) The value of maximal moment is determined by substituting the $x_e = 1$ m into the bending moment equation:

$$M_y^{II}(x_e) = -10x_e + 5x_e^2 = -10 + 5 = -5 \text{ kNm}.$$

Portion III – III $(0 < x < 2\ \text{m})$

$N_x^{III}(x) = -P + qa = -10 + 20 = 10$ kN,

$Q_z^{III}(x) = -X_1 + X_2 = -6.29 + 7.742 = 1.452$ kN,

$M_y^{III}(x) = -M + X_1(a + x) - X_2 x - Pa + qa^2/2 =$

$= -10 + 6.29(2 + x) - 7.742x - 20 + 20 = -10 + 12.58 + 6.29x - 7.742x =$

$= -1{,}452x + 2{,}58\big|_{x=0} = 2{,}58\big|_{x=2} = -0{,}314$ kNm (linear function).

Portion IV – IV $(0 < x < 2\ \text{m})$

$N_x^{IV}(x) = X_1 - X_2 = 6.29 - 7.742 = -1.452$ kN,

$Q_z^{IV}(x) = -P + qa = -10 + 20 = 10$ kN,

$M_y^{IV}(x) = +M - M + 2aX_1 - aX_2 - P(a - x) + qa(a/2 - x) =$

$= 6.29 \times 4 - 7.742 \times 2 - 10(2 - x) + 20(1 - x) = 25.16 - 15.484 - 20 + 10x + 20 - 20x =$

$= -10x + 9.676\big|_{x=0} = 9.676\big|_{x=2} = -10.324$ kNm.

(11) Designing of graphs of bending moment and also shear and normal force distributions in the equivalent system.

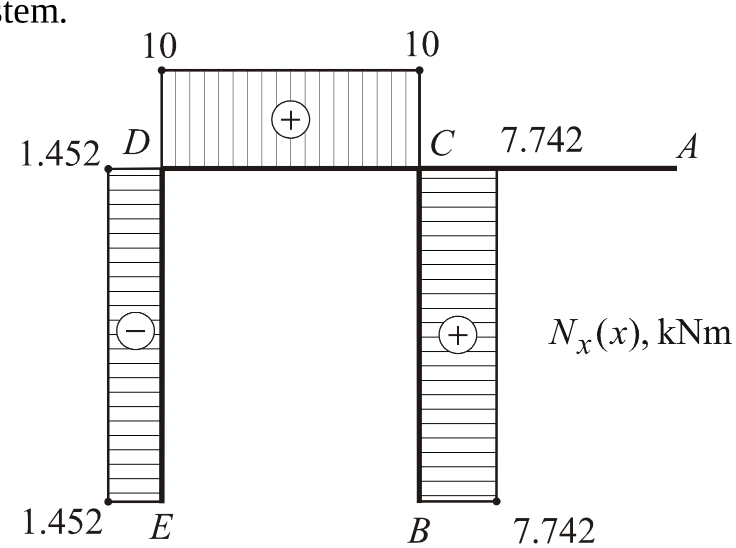


Fig. 6a

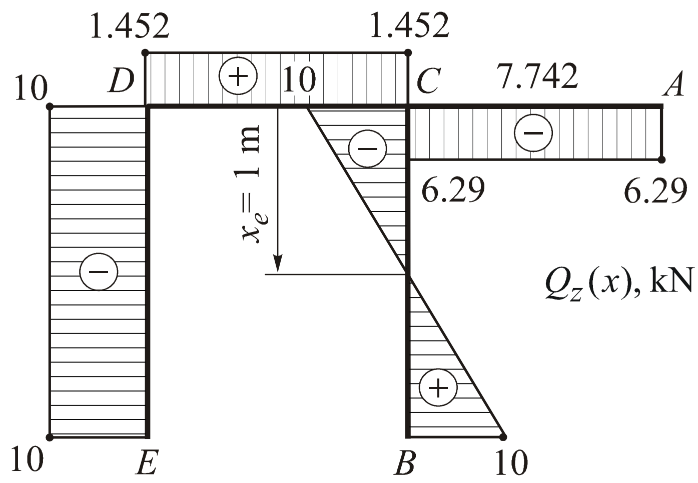


Fig. 6b

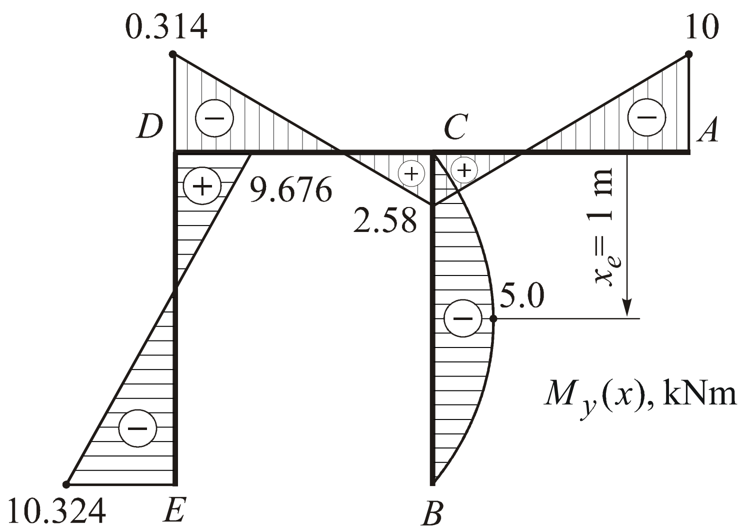


Fig. 6c

(12) Checking the equilibrium in two rod connections

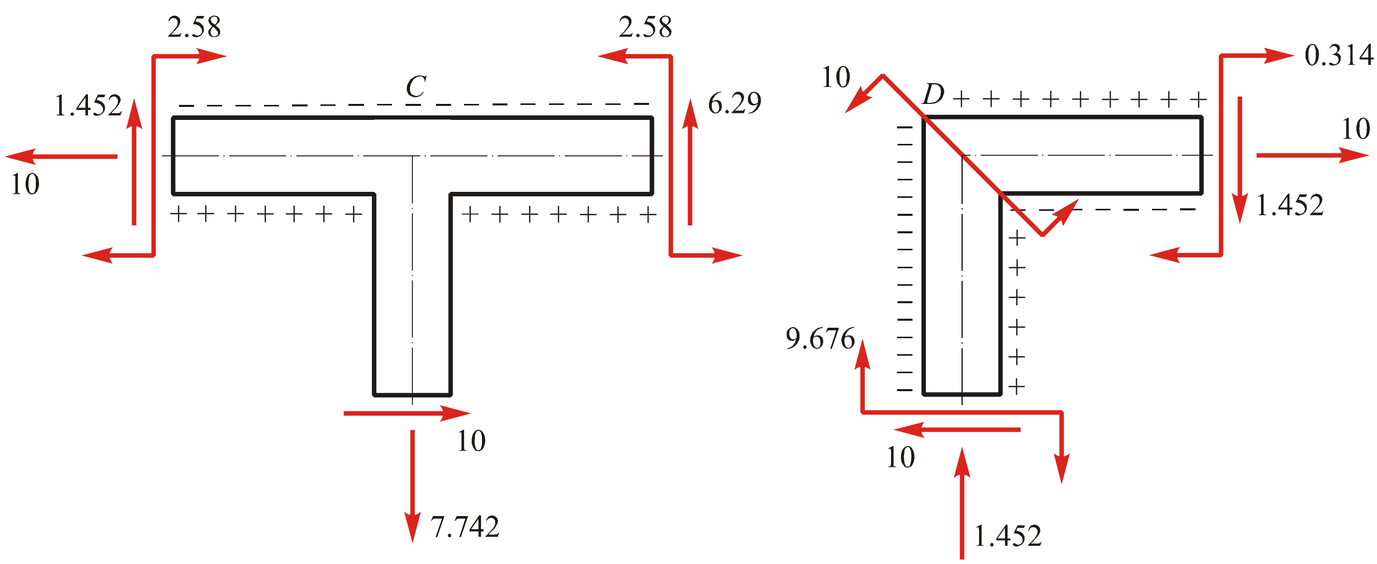


Fig. 7